Non Standard Foundations of Fuzzy Probabilities and Statistics of Vague Data

(PhD thesis by Panagiotis L. Theodoropoulos)

Abstract

The professor of University Patras Drossos C. had the idea that the transition from traditional models of Set Theory in non-classical would suggest and non-standard items whose logic would be many-valued. Thus, the fuzziness could be expressed better with the help of Non-Standard Analysis and the theory of Boolean Models in an original composition. For this he started with his associates a serious research. This research continued in this thesis, whose main purpose is the application of fuzziness, as amended by the new concepts in the Probabilities and Statistics, areas which are not dealt previous. Furthermore formulated and proved for the first time some propositions and theorems relating to the uncertainty as that formed here which we call IB-fuzziness.

Initially discussed the concept of IB-fuzziness in contradistinction to the classical fuzzy theory which introduced by Zadeh in 1965.

The basic idea of the definition of classical fuzzy sets has been the generalization of the characteristic function $I_A : X \to \{0, 1\}$ of a subset A of X (universe) to a function which has the form $f : X \to [0, 1]$. For the creation of the model IB-fuzziness, the set {0, 1} is treated as a trivial algebra Bool, since the operations between sets defined by respective logical operations and therefore the characteristic function I_A is generalized to a function which has the form $f : X \to IB$, where *IB* is a general complete Boolean algebra.

Furthermore, summarized the theory of Boolean models which are used in the thesis. The logic of Boolean models is the many valued. I.e. a logic sentence *p* can to take truth value, denoted by $\|p\|$, not only 0 or 1, but anything element of a general algebra Bool *IB*.

Then founded the IB-fuzzy real numbers discrete and non-discrete. Discrete IB-fuzzy real numbers are defined as the elements of the set:

$$IR[IB] = \left\{ f \in IB^{IR} : (\forall x, y \in IR)\left[x \neq y \to f(x) \wedge f(y) = 0_{IB}\right] \text{ και } \bigvee_{x \in IR} f(x) = 1_{IB} \right\}$$

If *r* is a relation two elements to the set *IR*, then this in the structure of *IR*[*IB*] is transferred as a many-valued relationship as follows:

$$\|r(f, g)\| := \bigvee_{\substack{x, y \in IR): \\ \mathrm{r}(x,\, y)}} \left(f(x) \wedge g(y) \right) \quad (1)$$

This creates the structure of the set *IR*[*IB*], which is said Boolean Power and of course this logic is many-valued.

For to be usable and quantitative informations on the expression of IB-fuzziness as algebra Boole IB we use the algebra Boole IB of a Probabilities algebra (*IB*, *p*), which satisfies the condition of countable chain. Therefore the values of an element of *IR*[*IB*] which are different of 0_{IB} are at most countable and thus created the set of discrete IB-fuzzy real numbers which is denoted by *IR*[*IB*]. By completeness of the set *IR*[*IB*] we take the non-discrete IB-fuzzy real numbers. The set of all IB-fuzzy real numbers denoted by the symbol $IR^{\#}$. Completing the foundation of IB-fuzzy real numbers we define the function of the standard part where each IB-fuzzy real number corresponding to a real number.

The core of the thesis is the **Boolean non-Standard Frame** creation gated within which is the study of the IB-fuzziness. All mathematics items related to the set of real numbers (sets, functions etc.) belong to the superstructure *V*(*IR*), which has a base set the *IR* and inductively defined as follows:

$$V_0(IR) := IR$$

$$V_{\kappa+1}(IR) := V_\kappa(IR) \cup P\big(V_\kappa(IR)\big),\ \forall \kappa \in IN$$

and

$$V(IR) := \bigcup_{n\in IN} V_n(IR)$$

The atomic types of superstructure *V*(*IR*) are the types of equality "=" and of belonging "$\in$" and the logic his is the two-valued logic.

Similarly, all IB-fuzzy mathematics objects associated with IB-fuzzy real numbers belong to the superstructure $V^{(IB)}(IR^{\#})$, which inductively defined as follows:

$$V_0^{(IB)}(IR^{\#}) := IR^{\#}$$

$$V_{n+1}^{(IB)}(IR^{\#}) := \left\{ f :\ Fun(f)\ \text{με}\ dom(f) \subseteq \bigcup_{\kappa=0}^{n} V_\kappa^{(IB)}(IR^{\#})\ \ \text{και}\ ran(f) \subseteq IB \right\}, \forall n \in IN$$

and

$$V^{(IB)}(IR^{\#}) := \bigcup_{n\in IN} V_n^{(IB)}(IR^{\#}).$$

The superstructure $V^{(IB)}(IR^{\#})$ is many-valued and the truth values for the atomic types of equality "=" and of belonging "$\in$" are defined as follows:

The equality in the set $IR^{\#}$ as in (1), while in $V^{(IB)}(IR^{\#})$ - $IR^{\#}$ with the relationship:

$$\|f=g\|:=\left(\bigwedge_{x\in dom(f)}\left(f(x)\Rightarrow\|x\in g\|\right)\right)\wedge\left(\bigwedge_{x\in dom(g)}\left(g(x)\Rightarrow\|x\in f\|\right)\right)$$

and of belonging "$\in$" as

$$\|x\in f\|:=\bigvee_{y\in dom(f)}\left(f(y)\wedge\|x=y\|\right)$$

The superstructure $V(IR)$ is embedded elementary in the superstructure $V^{(IB)}(IR^{\#})$ with main feature the **Transfer Principle** where transferred types and sentences from the superstructure $V(IR)$ in the superstructure $V^{(IB)}(IR^{\#})$.
Utilities are defined intermediate structure of Boolean Powers of all levels superstructure $V(IR)$. That is the total of:

$$W:=\bigcup_{n\in IN}V_n(IR)[IB]$$

with truth values for atomic types:

$$\|A=B\|:=\bigvee_{\substack{x\in dom(A),y\in dom(B):\\ x=y}}\left(A(x)\wedge B(y)\right)$$

and

$$\|x\in A\|:=\bigvee_{\substack{y\in dom(x),Z\in dom(A):\\ y\in Z}}\left(x(y)\wedge A(Z)\right)$$

The superstructure $V(IR)$ is embedded elementary in W structure, namely whether,

$$i\ :\ V(IR)\ \rightarrow\ W$$

is the immersion, then for each type $\varphi(x_1,\ ...,\ x_v)$ of $V(IR)$ and α_1, …, α_v elements of $V(IR)$ the logic sentence $\varphi(\alpha_1,\ ...,\ \alpha_v)$ is true in $V(IR)$ if and only if the logic sentence $\varphi(i(\alpha_1),\ ...,\ i(\alpha_v))$ is true e in W (having truth value equal with 1_{IB}).

Also, it turns out that there is an embedding m_{IB} of the W structure in $V^{(IB)}(IR^{\#})$ superstructure, which keeps the truth values of atomic types (and induction of all types), i.e. equations apply:

$$\|x\in A\|=\|m_{IB}(x)\in m_{IB}(A)\|$$

and

$$\|A=B\|=\|m_{IB}(A)=m_{IB}(B)\|$$

The composition of embedding i with m_{IB} gives the blocked elementary embedding of superstructure $V(IR)$ to the superstructure $V^{(IB)}(IR^{\#})$, described above.

Finally, the Boolean non-Standard Frame created is used to study the IB-fuzziness in Probabilities and Statistics, since, according to the Transfer Principle, transferring formulas and theorems by the standards Mathematics to the non standards Mathematics expressed by the IB-fuzziness.