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ANALYZING THE ROLE OF SHAPES IN THE PROCESS OF WRITING PROOFS IN MODEL OF P-M COMBINATIONS

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Abstract. The figure, that a student draws, when attempting to write a proof, does not represent a unique shape; it represents rather many shapes. All these shapes are representations that result from the unique figure. Moreover, a plane geometrical figure consists of points, line segments, rays, and lines. We call these "figure components". We stress that it is not quite easy for senior high school students to recognize either all the representations that result from a unique static figure or the roles that figure components play in a geometric sketch. In this article we analyse the role of shapes in the process of writing proofs in Model of p-m Combinations focusing both on three extremely important questions that help students geometry thinking and on the representations that result from dynamic geometry software (DGS) environments.

ZDM Subject Classification: G43 *AMS Subject Classification*: 00A35.

Key words and phrases: Model of p-m Combinations, figure components, representation, dynamic geometry software.

Introduction

In an earlier research of ours we attempted to teach the proof in Geometry using ideas from two well known theories [9]. Particularly, we combined both the phases proposed by the Theory of van Hiele [18, 19] and the methods of Cognitive Apprenticeship [4, 7, 8] and we enriched these combinations with our own ideas so as to make our instruction coherent with Vygotsky's ideas [20, 21]. We used the term "Model of Phases-Methods Combinations" or "p-m Combinations" to describe this model. We have also described a special worksheet, labeled "Structured Form Worksheet" (SFW) that we first coined and used to implement the above combination [9]. Additionally, we described an important component of SFW [10], called "Reasoning Control Matrix for the Proving Process" (RECOMPP). Every proof is constituted by two components, according to the "Model of Phases-Methods Combinations". These components may be either a *statement* and a *justification* (*of this statement*), or a *statement* and a *justification that corresponds to a partial proof*. We defined as "*simple justification*" or "*simple proof*", the case when the parts of the proof are exactly a *statement* and the *justification* (of this statement), and as a "non simple justification" the case when the parts of the proof are a *statement* and a *partial proof*. Therefore, we claim that *a proof can be analyzed to a simple justification or a partial proof*. Also, we have defined as "*simple proposition*", the one whose proof is a simple proof [9, 10, 16].

According to p-m Combinations model the instruction takes place in five periods (Fig. 1). During the first period, students relate the visual geometric shapes and their appearance with their names for every cognitive subject, e.g. of all kinds of parallelograms and their appearance with their names. Moreover the teacher increasingly demonstrates more complex shapes. The students acquaint with more complex shapes and their components.

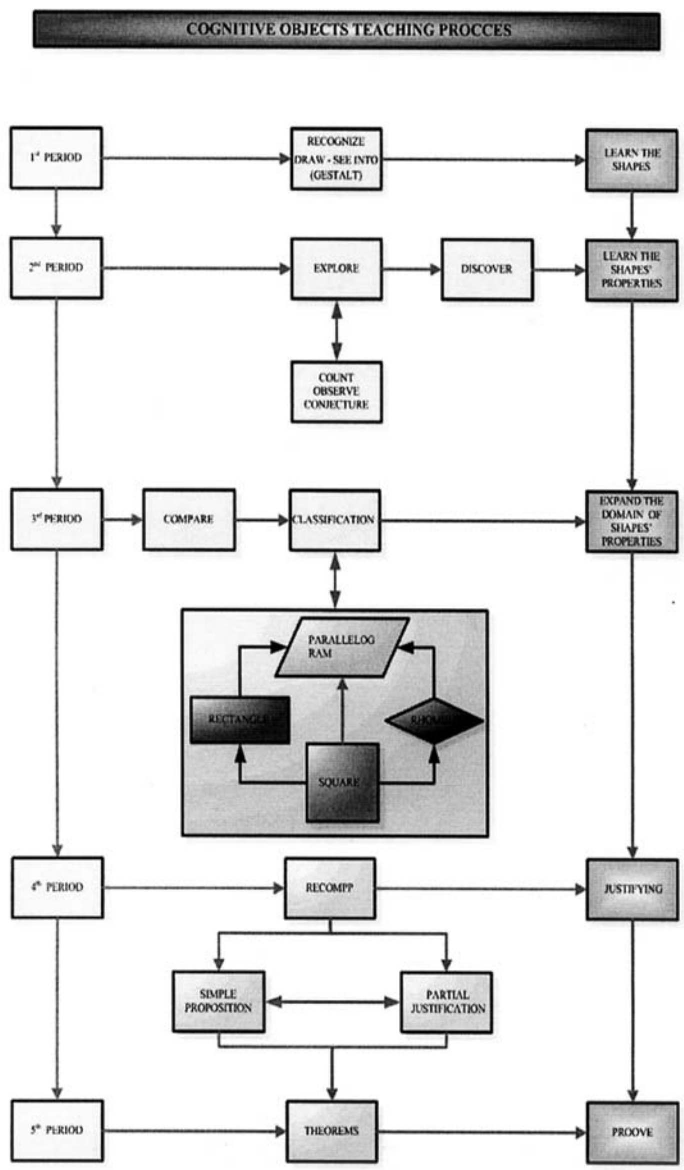

Fig. 1

During the second period, students are taught the attributes and the relative theorems of the cognitive subject (parallelograms), without their proofs. Students confirm the validity of theorems in an experimental way, e.g. using computers. During the third period, students classify the shapes (all kinds of parallelograms) and expand their shapes' properties. During the fourth period, students deal with the above mentioned simple (geometric) propositions and partial proofs. They learn to use RE.CO.M.P.P to write the proof. During the fifth period, students learn how to prove geometric propositions and they are taught the proofs of all theorems (of parallelograms). In this article we focus on the roles that the components of a figure play in the process of writing proofs in model of p-m Combinations.

The role of representations

The role of the figure is very important from the perspective of external representations. An internal representation is a hypothesised mental construct, but an external representation is a material notation of some kind, such as a graph, an equation, a geometrical, or a geometrical figure [1]. Nevertheless, the figure, that a student draws, when attempting to write a proof, represents more than one shape i.e. the external representations that result from a unique static figure are more than one. For example, when student wants to calculate the degree of angle BDC, given that we know the angles B and C (Fig. 2), he has to use not only the triangle ABC but also he needs to use the triangle DBC.

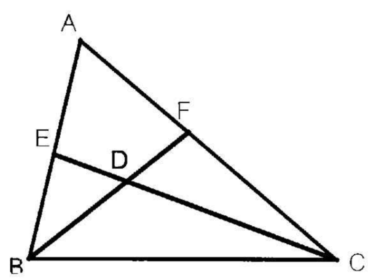

Fig. 2

More analytically, he has to use the triangle ABC three times (Fig. 3). One using the angle B, one using the angle C, etc.

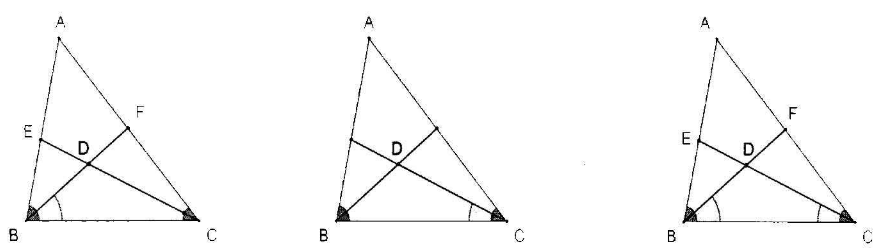

Fig. 3

The same has to be done with triangle DBC (Fig. 4). Every time the new representation of the same triangle (e.g. triangle ABC) is different from the previous one. The fact that different external representations result from the same figure is due to the fact the student works on different geometric elements and ideas in the unique shape.

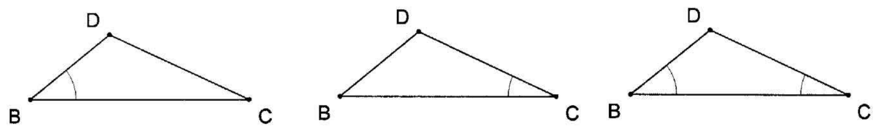

Fig. 4

In particular dynamic representations that result from dynamic geometry software (DGS) environments (particularly Geometer's Sketchpad and Cabri II+) play a semantic role as they aim to develop spatial sense and geometric reasoning. Moreover, DGS environments allow geometric postulates to be tested, offering "intelligent" constructivist tools that constrain users to select, construct, or manipulate objects that obey mathematical rules [12, 14, 15] and generally computer-based learning environments promise much in terms of enhancing mathematics learning [5, 13].

In our case we stress that using DGS environment we can help students to discriminate all the above mentioned required representations that help students to write the proof.

Figure components

We must stress that it is not quite easy for senior high school students to recognize either all representations that result from a unique static figure or the roles that every point and segment play in a geometric sketch. In particular a plane geometrical figure consists of points, line segments, rays, and lines. We call these "figure components". For example: a bisector of an angle is *a ray* (that cuts the angle exactly in half, making two equal angles), a triangle consists of three *segments*, and the centroid of a triangle is the *point* where the three medians meet, etc.

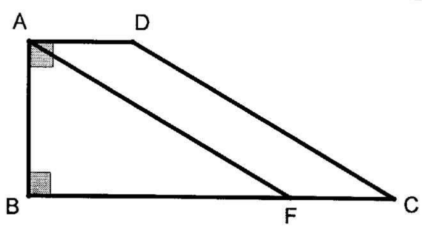

Fig. 5

But often a figure component, e.g. a line segment "plays" more than one roles in the same figure. In Fig. 5 the segment AF is the hypotenuse of the right triangle ABF, and it is also one of the sides of the parallelogram $AFCD$. Thus, the segment AF is provided with the properties of the hypotenuse of a right triangle and the properties of the sides of the parallelograms. The questions *What is it? What do I know about it? Does it remind me of something?* can help students to recognize the shapes and their components, but students in different van Hiele's levels give different answers. Thus, these questions play different roles during the five periods of instruction with the model p-m Combinations.

Analyzing the roles of the figure components

We can say, analyzing the roles of the segment AF in Fig. 5, that a student, who thinks at the *Recognition* (first level) according to van Hiele theory, cannot recognize all these properties. On the contrary, a student can see the shapes in a holistic way. Moreover he sees rather a right triangle and a parallelogram than a trapezoid, when considering the Fig. 5 and he thinks at the first level according to van Hiele theory [19]. Students see neither the double role that the segment AF plays nor the properties that arise from these roles. Of course, the answers to above questions have a holistic character. According to Model of Combinations in the 1st period the teacher must help students to make figures clear. So the teacher must provide students with several separate figures to help them recognize figures, learn their names and draw them correct. Also according to Model of Combinations in the first period the teacher must use appropriate examples to help students to make misconceptions clear. Moreover we claim that teachers have the potential to help students override some obstacles as defined by Bachelard [2], and referred to by Brousseau [3] by deploying dynamical representations of ICT in the teaching-learning process. For example the students do not always recognize whether two lines are perpendicular. Also, they do not always recognize the right angles. The orientation of a right angle affects the possibility that students recognize it, i.e. the successfully recognise a right angle it depends on the orientation of the right angle. Especially, they recognize the right angle in Fig. 6 but they do not recognize the right angle in Fig. 7.

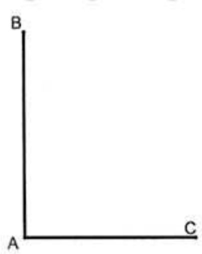

Fig. 6

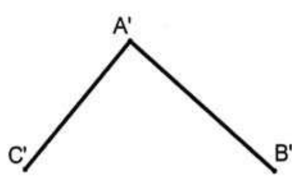

Fig. 7

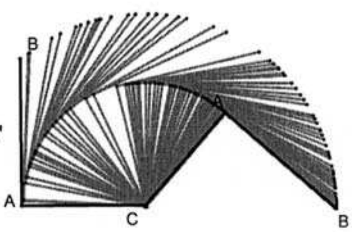

Fig. 8

In the case of the angles mentioned above teacher rotates the angle CAB by deploying dynamical representations of ICT, and fit in with the angle $C'A'B'$ (Fig. 8). In this way the teacher causes in the student a cognitive conflict. In particular as the learner restructures his mathematical schema to understand the new idea, cognitive conflict is bound to occur [17].

The *Analysis* of van Hiele theory must be understood as a means of helping students to discriminate the roles of a figure component e.g. students recognize the roles the segment AF plays, but they are not yet able to connect and combine the required properties, theorems, axioms, and the definitions to construct a complete proof. In our research the way students see the shape in this level allows them to make simple comments only in one particular figure. So, in this level the role of the line segments is very important.

According to Model of Combinations in the 2nd period students must examine every segment of the figure very carefully. Because of the above they must observe the ends of the line segments very carefully. For example the points A and M are the ends of the segment AM. The questions *What is it? What do I know about it? Does it remind me of something?* i.e. questions that refer to the components of figure, e.g. a question that refers to line segment, helps students to recognize its multiple roles in the figure. They are taught to use these phrases in the second period of Model of Combinations. Especially, in Fig 9 a student can ask himself: what is AM? If he observes the ends of the segment AM he will answer that AM is the median that joins the vertex of right angle with the midpoint of the hypotenuse. For the question *what do I know about it* (the median that joins the vertex of right angle with the midpoint of the hypotenuse), he will answer $AM = \frac{1}{2}\,BC$. Thus in Analysis the questions *what is it? What do I know about it? Does it remind me of something?* are extremely important.

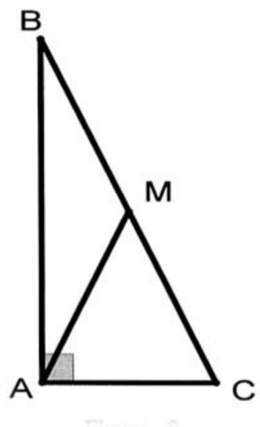

Fig. 9

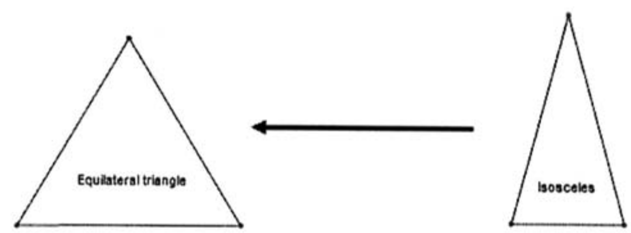

Fig. 10

The *Ordering* of van Hiele theory must be understood as a mean of transferring the properties from one figure to the second one, given that both the first and second figures belong to the same category. For example, the properties of an isosceles triangle are part of the properties of an equilateral triangle, but the converse is not true (Fig.10). Similarly, the properties of a parallelogram are part of the properties of a square, but the converse is not true (Fig. 11). So, if we see the ordering from the *3rd period of Model of Combinations perspective*, we can facilitate students learn geometry employing an easier way than that of the traditional methods. This can happen, because while students learn the properties of the parallelograms

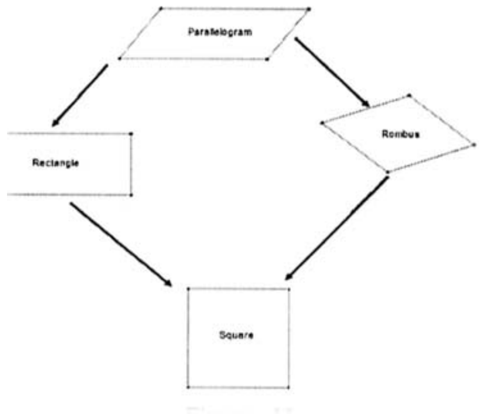

Fig. 11

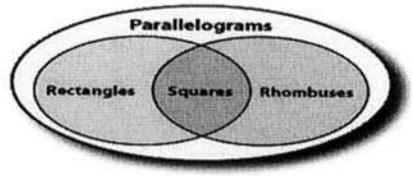

Fig. 12

(rectangles, rhombuses) they simultaneously learn a part of properties of the square (Fig. 12).

We believe that before students learn how to write formal proofs they should be able to write simple proofs. So, from the *fourth period of Model of Combinations perspective*, the questions *what is it? What do I know about it? Does it remind me of something?* play more important role than the one in the periods until now. These questions help students to discriminate the shapes in more details, especially their properties. Thus, although the concept neither of definition nor of theorems are not yet clarified, students are just able to make *simple proofs* (as defined above). At this moment of instruction, we propose the use of RECOMPP, because it urges students to combine the required properties, theorems, axioms, and the definitions for constructing *only simple justifications*, i.e. they make simple proofs but not complete proofs yet. For example, students may use of the Pythagorean Theorem for the right triangle ABF (Fig. 5), i.e. they are able to write the relationship, and they understand the following equalities:

- $AF = CD$, because AF and CD are the opposite sides of the parallelogram $AFCD$, and
- $BF = BC - FC$, as the result of subtracting segment FC from BC.

Nevertheless, they are not capable of proving the equality $AB^2+(BC-FC)^2 = CD^2$ or a more complex one that represents a combination of the above mentioned relationships.

At the fourth level *Deduction* students are able to see all the components consisting a geometrical figure, and they recognize all the roles a component plays. Specifically, students can analyze the geometrical figures in their components and in their roles helped by the questions *what is it? What do I know about it? Does it remind me of something?* From the Combinations Model perspective the fifth period is the period that students are taught how to prove.

The importance in this period is that the students have already known and have already used the theorems that they will prove. This happened in the earlier periods. So, students have acquired enough experience of these theorems. Not only have they understood the notions of the theorems, but also they have used them in simple proofs. This means that students have to learn only the proof of theorems and this is the objective target of Combinations Model.

From all the above emerges that these three questions play different roles in the five periods of instruction (Table 1).

Table -1: Students' answers in the periods

	1st Period	2st Period	3st Period	4st Period	5st Period
What is it?	Recognize. Shape recognition (holistically)	Parts of the shapes	Classifications of shapes	Shapes and their parts	Combination of shapes and their parts
Does it remind me of something?	Shape recognition from real models Definitions	Descriptions of the parts of the shapes	Discrimination of the shapes that belong in the same classification	Relationships among the figure components	Combination of relationships among the figure's components
What do I know about it?	Shape Definitions	Properties propositions and theorems of the parts of shapes	Classification of properties, propositions and theorems of the shapes	Relationships between Propositions and Theorems	Relationships among properties, propositions and theorems

Discussion

According to above analysis there is close correspondence between the periods of instruction of p-m Combinations Model and van Hiele levels (Fig. 13), but we stress that we have related two periods in the fourth level. This is our interference on the proving process. We believe that before students learn how to write formal proofs they should be able to write simple and partial proofs.

The questions *What is it? What do I know about it? Does it remind me of something?* can help students to recognize the shapes and their components. Thus, these questions play different roles along the five periods of instruction. The teacher must help students to learn how to use these questions in each one of these periods.

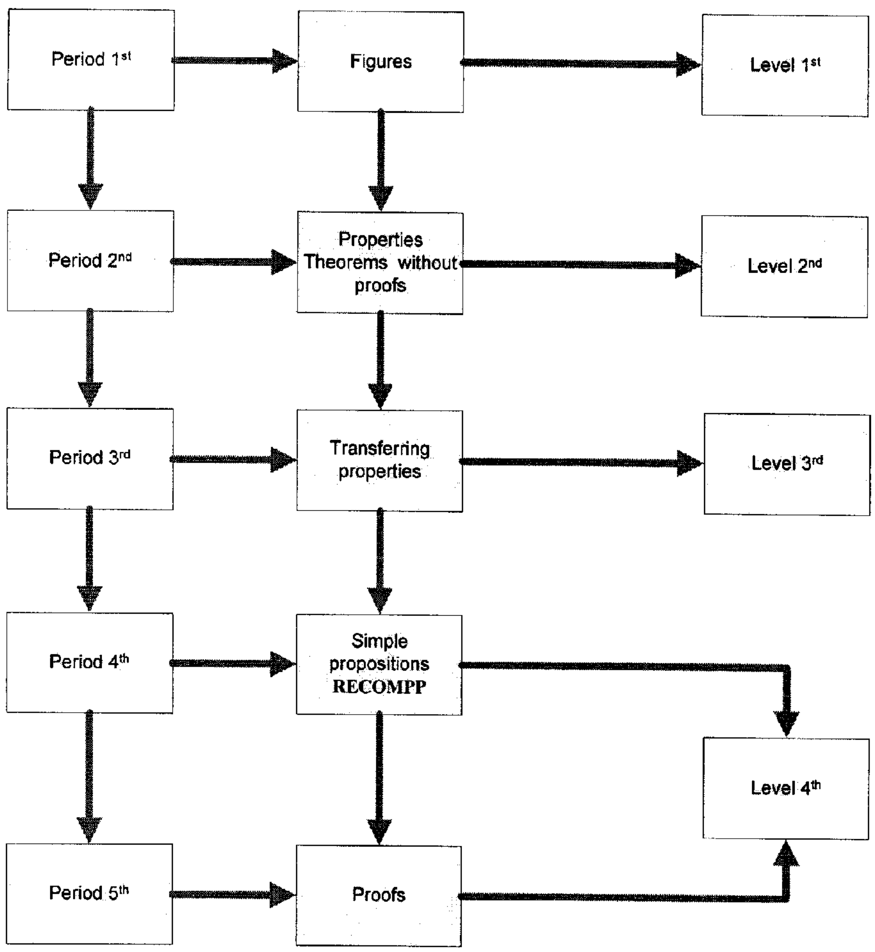

Fig. 13

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