The Kepler Problem: the Newtonian and the relativistic (Special Theory) point of view

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Synopsis

In this work we solve the Kepler problem in the context of two theories: a) the Special Theory of Relativity and b) the Newtonian Mechanics. We simulate the motion of a particle according to the predictions of each model, for the same initial conditions. We observe and calculate the precession of the perihelion of the orbit for the case of the relativistic model and the closed-elliptic path in the Newtonian model. We check our calculations and the virtual measurements, by implementing several activities in the environment of the simulation.

Open the simulation

Key concepts and relations

Minkowski space - The Kepler problem - Minkowski force - Minkowski equations of motion - Perihelion - Aphelion - Precession of aphelion or of perihelion - The Newtonian model for the Kepler problem

Contents

1. A generalization of the Kepler force acting on a particle P in the context of the Special Theory of Relativity

Remarks on symbolism

a) Greek indices take the values 0,1,2,3 and the Latin 1,2,3
b) We use the symbolism $x = \left(x^0, x^1, x^2, x^3\right)$
c) We use alternatively the symbols: $ct = x^0, x = x^1, y = x^2, z = x^3$; $v_x = v^1, v_y = v^2, v_z = v^3$
d) The dot over the symbol of a quantity means derivative over time, and the primed symbol means derivative over the polar angle θ. For example: $\dot{r} = \dfrac{dr}{dt}$ but: $r' = \dfrac{dr}{d\theta}$; it holds: $\dot{r} = r'\dot{\theta}$

The particles' equations of motion in a Minkowski space arise as a generalization of the 2nd Newton's law [1,2,4]. This generalization leads to the "Minkowski equation of motion" which is invariant under any coordinate transformation in M. We write:

$$\frac{DP}{D\tau} = K \text{ or: } \frac{DP^\mu}{D\tau} = K^\mu \qquad (1.1)$$

Where: K is the Minkowski-4-force, τ is the proper time of the moving particle P and P the four-momentum of P: $P = mU$, and U is the 4-velocity of P. *K, P* and *U* are vector fields in *M*, which run the tangent vector spaces $T_X M$, $X \in M$ of *M*. It holds:

$$T_X M \ni U = e_\mu U^\mu = e_\mu \frac{dx^\mu}{d\tau} = e_\mu \gamma \frac{dx^\mu}{dt} = \left(e_0 + e_j \frac{v^j}{c}\right) c\gamma$$

$\gamma^{-1} = \sqrt{1-\frac{v^2}{c^2}}$, $v^2 = (v^1)^2 + (v^2)^2 + (v^3)^2$

e_μ, $\mu = 0,1,2,3$ are the basis vectors in each tangent vector space of the Minkowski space, in the inertial Cartesian x-coordinate system of reference [2,4].

The Minkowski force is always orthogonal to the 4-velocity of the particle [2,4]:

$$\left\langle \frac{DP}{D\tau}, U \right\rangle = \langle K, U \rangle = 0,\ K^0 c - \sum_{j=1}^{3} K^j v^j = 0 \qquad (1.2a)$$

Define $K_\mu = g_{\mu\nu} K^\nu$, $K_0 = K^0$, $K_j = -K^j$ Then:

$$K_0 c + K_j v^j = 0 \qquad (1.2b)$$

The solution of the differential equations 1.1, under the constraint 1.2a or b, determines the world line of the moving particle. The appropriate free parameter to express the analytic form of the world line is the world time t [3,4]. From 1.1 we obtain:

$$\gamma \frac{d}{dt}\left(m\gamma c\right) = K^0 \text{ (a)},\ \gamma \frac{d}{dt}\left(m\gamma v^j\right) = K^j \text{ (b)} \qquad (1.3)$$

An acceptable form of the Minkowski force, which satisfies condition 1.2a is the following [2,4]:

$$K = e_\mu K^\mu,\ K^\mu = -\frac{1}{c^2} V(\mathrm{x}) \frac{dU^\mu}{d\tau} + \partial_\nu V(\mathrm{x}) \left(g^{\mu\nu} - \frac{1}{c^2} U^\mu U^\nu \right) \qquad (1.4a)$$

$V(\mathrm{x})$ is a differentiable scalar function and $\left[g^{\mu\nu}\right]$ the inverse of the metric tensor:

$$\left[g^{\mu\nu}\right] = \left[g_{\mu\nu}\right]^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Assume that in the considered Cartesian inertial frame, $V(\mathrm{x})$ is independent of the time coordinate: $\partial_0 V(\mathrm{x}) = 0$. Then, equations 1.3a and b take the form:

$$(1.3a) \Rightarrow \frac{d}{dt}\left(\gamma \left(1 + \frac{V(\mathrm{x})}{mc^2} \right) \right) = 0 \qquad (1.5a)$$

$$(1.3b) \Rightarrow m\gamma \frac{d}{dt}\left(\gamma v^j \left(1 + \frac{V(\mathrm{x})}{mc^2} \right) \right) = -\partial_j V(\mathrm{x}) \qquad (1.5b)$$

From 1.5a we imply the energy conservation along the path:

$$\gamma \left(1 + \frac{V}{mc^2} \right) = \frac{E}{mc^2} \underset{def}{=} \varepsilon = \text{constant} \qquad (1.6)$$

1.5a and 1.5b are written:

$$\gamma \left(1 + \frac{V(\mathrm{x})}{mc^2} \right) = \varepsilon \qquad (1.7a)$$

$$\gamma \frac{dv^j}{dt} = -\frac{1}{\varepsilon m} \partial_j V \qquad (1.7b)$$

For $\frac{v^2}{c^2} << 1$ and $\frac{|V(\mathrm{x})|}{mc^2} << 1$ 1.7a and 1.7b converge to the equations:

$$mc^2 + \frac{mv^2}{2} + V(\mathrm{x}) = E \qquad (1.8a)$$

$$m \frac{dv^j}{dt} = -\partial_j V \qquad (1.8b)$$

The approximate equations 1.8a and b can be considered as the differential equations of the motion for a particle in the Newtonian context, if we identify the function $V(\mathrm{x})$ with the potential energy of the particle, in the specific Cartesian inertial frame.
Assume that in this specific inertial frame, in the Newtonian limit the particle is moving under the action of the Newtonian gravitational field of forces: $F_j = -\frac{GMm}{r^3}x^j$. Then, $V(\mathrm{x})$ is determined by the Newtonian gravitational potential energy:

$$V(\mathrm{x}) \equiv V(r) = -G\frac{Mm}{r},\ r = \left((x^1)^2 + (x^1)^2 + (x^1)^2\right)^{1/2}$$

G is the gravitational constant, M the mass of the attracting point which is fixed at the spatial origin O of the coordinate system, and m the mass of the moving particle P.

We conclude that the Kepler path of P in the context of the Special Theory of Relativity is a solution of the equations:

$$\gamma\left(1 - \frac{GM}{c^2}\frac{1}{r}\right) = \varepsilon = \text{const.} \qquad (1.9a)$$

$$\gamma\frac{dv^j}{dt} = -\frac{GM}{\varepsilon}\frac{1}{r^3}x^j \qquad (1.9b)$$

In what follows, we confine our task in the case that ε is positive:

$$\varepsilon > 0 \text{ or: } r - \frac{GM}{c^2} > 0 \qquad (1.10)$$

$$\int_a^x \omega_p\Big|_c = \int_a^x \omega_p\Big|_{c'}$$

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2. The particle's orbit according to the Special Theory of Relativity

We transform differential equations 1.9, so that being able to formulate arithmetically a unique solution, under some specific initial conditions.
The quantities x^j, v^j are determined in the Euclidean space E_3 of the coordinates. Hence 1.9 can be written in vector form:

$$\gamma\left(1 - \frac{GM}{c^2}\frac{1}{r}\right) = \varepsilon \qquad (2.1a)$$

$$\gamma\frac{d\vec{v}}{dt} = -\frac{GM}{\varepsilon}\frac{1}{r^3}\vec{r} \qquad (2.1b)$$

Multiply externally both sides of 2.1b with $\vec{r}$:

$$2.1b \Rightarrow \vec{r}\times\frac{d\vec{v}}{dt} = 0 \Rightarrow \frac{d}{dt}\left(\vec{r}\times\vec{v}\right) = 0$$

We imply that the quantity: $\vec{l} = m\vec{r}\times\vec{v}$ is conserved along the path drawn by the particle P; let us name it "Newtonian angular momentum" (NAM) of P.
Assume that the initial conditions of the moving particle are specified by the expressions:

$$\vec{r}(0) = \left(x_0, 0, 0\right) = \hat{x}x_0,\ \vec{v}(0) = \left(0, v_0, 0\right) = \hat{y}v_0 \qquad (2.2a)$$

The constant value of NAM is:

$$\vec{l} = m\vec{r}\times\vec{v} = \hat{x}\times\hat{y}\,mx_0v_0 = \hat{z}\,mx_0v_0 \qquad (2.2b)$$

2.2b implies that the orbit $\vec{r} = \vec{r}_P(t)$ lies on the Oxy plane and that the norm of the external product $\vec{r}\times\vec{v}$ is a constant of the motion. The following conservative equation holds along the path of P:

$$\vec{l} = \hat{z}\,m\left(xv_y - yv_x\right) = \hat{z}mx_0v_0 = \text{const.} \qquad (2.2c)$$

Hence, equations 2.1a and b, are simplified to the next:

$$\frac{d\vec{v}}{dt} = -\frac{1}{\varepsilon^2}\left(1 - \frac{GM}{c^2}\frac{1}{r}\right)\frac{GM}{r^3}\vec{r} \qquad (2.3a)$$

$$\vec{r} = \left(x, y, 0\right), \ \vec{v} = \left(v_x, v_y, 0\right), \ \varepsilon^2 = \frac{\left(1 - \dfrac{GM}{c^2 x_0}\right)^2}{1 - \dfrac{v_0^{\,2}}{c^2}}, \ r = \sqrt{x^2 + y^2} \qquad (2.3b)$$

According to the constrain 1.10:

$$\varepsilon > 0, \ x_0 > \frac{GM}{c^2} \qquad (2.3c)$$

The equations of motion in polar coordinates

We formulate the initial conditions and the equation of the motion in polar coordinates:

$x = r\cos\theta$, $y = r\sin\theta$; $v_x = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$, $v_y = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$; $\gamma^{-1} = \sqrt{1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}$

The initial conditions take the form:

$$\theta(0) = 0, \ r(0) = x_0, \ \dot{r}(0) = 0, \ \dot{\theta}(0) = \frac{v_0}{x_0} \qquad (2.4a)$$

$$\gamma(0) = \frac{1}{\sqrt{1 - \dfrac{v_0^{\,2}}{c^2}}}, \ \varepsilon = \gamma(0)\left(1 - \frac{GM}{c^2 x_0}\right) \qquad (2.4b)$$

From 2.2b we find that:

$$r^2\dot{\theta} = \frac{l}{m} = x_0 v_0 \ \text{or:} \ \dot{\theta} = \frac{x_0 v_0}{r^2} \qquad (2.5)$$

The energy conservation is expressed as follows:

$$\varepsilon = \gamma\left(1 - \frac{GM}{c^2}\frac{1}{r}\right) = \frac{1 - \dfrac{GM}{c^2}\dfrac{1}{x_0}}{\sqrt{1 - \dfrac{v_0^{\,2}}{c^2}}} > 0 \qquad (2.6a)$$

$$\gamma^{-1} = \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - \frac{1}{c^2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)} = \\ = \sqrt{1 - \frac{x_0^{\,2} v_0^{\,2}}{c^2}\frac{1}{r^4}\left(r'^2 + r^2\right)}, \ \text{where:} \ r' \underset{def}{=} \frac{dr}{d\theta} \qquad (2.6b)$$

We define the constants:

$$R_0 = \frac{x_0 v_0}{c}, \ R_1 = \frac{GM}{c^2} \qquad (2.7)$$

…which both have dimensions of length.

From 2.6a, 2.5 and 2.7a, we find out the equation of the path in polar coordinates:

$$\frac{dr}{d\theta} = \frac{r}{\varepsilon R_0} f(r), \ \frac{d\theta}{dt} = \frac{x_0 v_0}{r^2} \qquad (2.8)$$

…where: $f(r) = \sqrt{-\left(1 - \varepsilon^2\right)r^2 + 2R_1 r - \left(R_1^{\,2} + \varepsilon^2 R_0^{\,2}\right)}$

We focus on the case that the orbit is bounded in a compact region of the Oxy plane [(1)]. We can show that this is true for $\varepsilon^2 < 1$:

We can easily prove that x_0 is a root of the equation:

$$\left(\varepsilon^2-1\right)x_0^{\,2}+2R_1x_0-\left(R_1^{\,2}+\varepsilon^2R_0^{\,2}\right)=0 \qquad (2.8a)$$

Hence, the second order polynomial equation:

$$\left(\varepsilon^2-1\right)r^2+2R_1r-\left(R_1^{\,2}+\varepsilon^2R_0^{\,2}\right)=0 \qquad (2.8b)$$

must have two real roots. This is true if only its discriminant is not negative:

$$R_1^{\,2}-\left(1-\varepsilon^2\right)\left(R_1^{\,2}+\varepsilon^2R_0^{\,2}\right)\geq 0 \Rightarrow R_1^{\,2}\geq R_0^{\,2}\left(1-\varepsilon^2\right) \qquad (2.8c)$$

If in addition it is true that $1-\varepsilon^2>0$, the polynomial has two positive roots, and it takes positive values if only *r* runs the interval between the roots. Hence, the trajectory is bounded; it is confined in a compact region of the Oxy plane.

By using 2.6a, we express ε^2 as a function of the parameters R_0, R_1 and x_0: $\varepsilon^2=\dfrac{\left(x_0-R_1\right)^2}{x_0^{\,2}-R_0^{\,2}}$

Hence, the roots of $f^2(r)$ are identical to the solutions of the equation:

$$\left(-2x_0R_1+R_0^{\,2}+R_1^{\,2}\right)r^2+2R_1\left(x_0^{\,2}-R_0^{\,2}\right)r-x_0^{\,2}\left(R_0^{\,2}+R_1^{\,2}\right)+2x_0R_1R_0^{\,2}=0 \qquad (2.8d)$$

We can immediately verify that x_0 is a solution of 2.8b. Its second solution is calculated by the expression:

$$r_2=\frac{x_0\left(R_0^{\,2}+R_1^{\,2}\right)-2R_1R_0^{\,2}}{2x_0R_1-\left(R_0^{\,2}+R_1^{\,2}\right)} \qquad (2.8e)$$

For $r=x_0$ and for $r=r_2$ it holds: $\dfrac{dr}{d\theta}=0$ (see equations 2.8); x_0 and r_2 correspond to the extrema of the orbit. We call "perihelium" the $\min\left(x_0,r_2\right)\underset{def}{=}r_p$ and "aphelion" the $\max\left(x_0,r_2\right)\underset{def}{=}r_a$.

For $r\in\left[r_p,r_a\right]$, $f(r)$ takes the form:

$$f(r)=\sqrt{-\left(1-\varepsilon^2\right)\left(r-r_p\right)\left(r-r_a\right)}=\sqrt{\left(1-\varepsilon^2\right)\left(r-r_p\right)\left(r_a-r\right)} \qquad (2.8f)$$

Circular orbit in the relativistic model

The conditions which must be satisfied for the particle to realize a circular orbit, are a) equation 2.8b has a double solution, i.e., $r_2=x_0$ and b) $r=x_0$ for every time *t*.

Which is the value of the initial velocity $\left(0,v_0,0\right)$ for which the circular orbit is obtained?

From 2.8c and condition (a), it is implied that the requested value of v_0 is:

$$v_0=\sqrt{\frac{GM}{x_0}} \qquad (2.9a)$$

The external energy of P, in this case, is calculated by the subsequent equations (see 1.9 and 2.9a):

$$E_{cR}=mc^2\left(\varepsilon-1\right)=mc^2\left(\sqrt{1-\frac{GM}{c^2x_0}}-1\right)\approx -m\frac{GM}{2x_0}-m\frac{\left(GM\right)^2}{8c^2x_0^{\,2}}-\dots \qquad (2.9b)$$

Precession of the perihelion of the orbit in the relativistic model

What is the angle $\Theta_{p\to a}$ drawn by the position vector $\overrightarrow{OP}$ when P is moving from the perihelion to the aphelion and what is the needed time?

From 2.8a, we obtain:

$$\Theta_{p\to a}=\varepsilon R_0\int_{r_p}^{r_a}\frac{dr}{r\,f(r)} \qquad (2.10a)$$

…where $f(r)$ is given by 2.8d, and $\varepsilon>0$, according to 1.10.

Approximate calculation of the integral in 2.10a.
Consider a partition of the interval $\left[r_p, r_a\right]$:

$$\left[r_p, r_a\right] = \left[r_p, r_1\right] \cup \left[r_1, r_2\right] \cup \ldots \left[r_{N-1}, r_N\right],\ r_N = r_a$$

Where, N is an integer and: $r_n = r_p + nDr$, $n = 0,1..N$, $Dr = \left(r_a - r_p\right)/N$
In the k-interval of the partition pick the point:

$$q_k = r_k + \frac{Dr}{2} = r_p + Dr\left(k + \frac{1}{2}\right) = r_a + Dr\left(k + \frac{1}{2} - N\right), k = 0,1..N-1$$

Then, the integral in the right part of 2.10 is approximated by the expression:

$$\int_{r_p}^{r_a} \frac{dr}{r\,f(r)} \approx \sum_{k=0}^{N-1} \frac{Dr}{q_k f\left(q_k\right)} =$$

$$= \sum_{k=0}^{N-1} \frac{Dr}{q_k \sqrt{\left(1-\varepsilon^2\right)\left(q_k - r_p\right)\left(r_a - q_k\right)}} =$$

$$= \frac{1}{\sqrt{1-\varepsilon^2}} \sum_{k=0}^{N-1} \frac{1}{\left(r_p + Dr\left(k+\frac{1}{2}\right)\right)\sqrt{\left(k+\frac{1}{2}\right)\left(N-k-\frac{1}{2}\right)}}$$

We conclude that:

$$\Theta_{p\to a} \approx A \sum_{k=0}^{N-1} \frac{1}{\left(r_p + Dr\left(k+\frac{1}{2}\right)\right)\sqrt{\left(k+\frac{1}{2}\right)\left(N-k-\frac{1}{2}\right)}} \qquad (2.10b)$$

where: $A = \dfrac{R_0 \varepsilon}{\sqrt{1-\varepsilon^2}} > 0$

The time needed for the particle to move from the perihelion to the next aphelion is calculated by equations 2.8, following the earlier procedure:

$$Dr = \frac{dr}{d\theta}\frac{d\theta}{dt}Dt = \frac{x_0 v_0}{\varepsilon R_0}\frac{f(r)}{r}Dt \Rightarrow Dt = \frac{\varepsilon R_0}{x_0 v_0}\frac{r}{f(r)}Dr$$

We conclude that:

$$T_{p\to a} \approx \frac{A}{x_0 v_0} \sum_{k=0}^{N-1} \frac{r_p + Dr\left(k+\frac{1}{2}\right)}{\sqrt{\left(k+\frac{1}{2}\right)\left(N-k-\frac{1}{2}\right)}} \qquad (2.10c)$$

To pass to the Newtonian limit without having to cope with indeterminate forms, we express the constants A and r_2 as follows:

$$A = R_0 \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = \left(x_0 - \frac{GM}{c^2}\right) \frac{x_0 v_0}{\sqrt{\left(x_0 v_0\right)^2\left(\frac{2GM}{x_0 {v_0}^2} - 1\right) - \frac{(GM)^2}{c^2}}} \qquad (2.10d)$$

$$r_2 = x_0 \frac{1 + \frac{2GM}{c^2 x_0}\left(\frac{GM}{2x_0 {v_0}^2} - 1\right)}{\frac{2GM}{x_0 {v_0}^2} - 1 - \frac{(GM)^2}{c^2 {x_0}^2 {v_0}^2}},\ Dr = \frac{1}{N}\left|x_0 - r_2\right| \qquad (2.10e)$$

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3. The Newtonian model

In the Newtonian limit, under the condition $1-\varepsilon^2>0$ and $\varepsilon>0$, the constants r_2 and A are figured out by approximating the expressions 2.10d, 2.10e. We obtain:

$$r_{2N}\approx\frac{x_0}{\frac{2GM}{x_0{v_0}^2}-1} \text{ and: } A_N\approx\frac{x_0}{\sqrt{\frac{2GM}{x_0{v_0}^2}-1}}=\sqrt{x_0r_{2N}} \qquad (3.1)$$

We infer that in the Newtonian model, the boundness of the trajectory is achieved under the condition: $\frac{2GM}{x_0{v_0}^2}-1>0$ or: $E_N=\frac{m{v_0}^2}{2}-\frac{GMm}{x_0}<0$ i.e., if the (external) energy of the moving particle is negative. In that case, the differential equation of the path is obtained by 2.8 and 3.1:

$$\frac{dr}{d\theta}=\frac{r}{\sqrt{x_0r_{2N}}}\sqrt{(r-x_0)(r_{2N}-r)} \qquad (3.2)$$

In Cartesian coordinates, the differential equations of the motion are derived from 2.3a:

$$\frac{d\vec{v}}{dt}=-\frac{GM}{r^3}\vec{r} \qquad (3.3)$$

Circular orbit in the Newtonian model

The conditions which must be satisfied to obtain circular orbit are:

a) $r_{2N}=x_0$ and b) $r=x_0$ for any value of *t*.

From 3.1 and condition (a), we imply that this is accomplished if the value of the initial velocity is:

$$v_0=\sqrt{\frac{GM}{x_0}} \qquad (3.4)$$

We see that 3.4 is identical to 2.9a which has been derived in the relativistic context. Nevertheless, the energy of P is different in the two models. In the Newtonian context the energy is:

$$E_{cN}=-\frac{GMm}{2x_0} \qquad (3.5a)$$

In the relativistic context the corresponding energy is given by 2.9b. We infer that:

$$E_{cR}=E_{cN}-m\frac{(GM)^2}{8c^2{x_0}^2}-\ldots<E_{cN} \qquad (3.5b)$$

If we demand that the initial energy of the circulating particle is the same in both models, the Newtonian radius r_N should be different of the relativistic radius r_R. From 2.9b and 3.5a, we obtain the subsequent relations:

$$E_{cR}=E_{cN}\Rightarrow-\frac{GMm}{2r_R}-m\frac{(GM)^2}{8c^2{r_R}^2}-\ldots=-\frac{GMm}{2r_N}$$

$$\frac{1}{r_R}+\frac{GMm}{4c^2{r_R}^2}+\ldots=\frac{1}{r_N}\Rightarrow\frac{r_R}{r_N}>1\Rightarrow r_R>r_N \qquad (3.6)$$

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4. The virtual environment of the simulation

Open the simulation

In the environment of the simulation, we set (in simulation-units):
$G = 0.01$ sim-units, $c = 1$ sim-units, $x_0 = 1$ sim-units (4.1)

The user controls the value v_0 of the initial velocity $(0, v_0, 0)$, and the mass M of the central point O which creates the gravitational field. In paragraph 2, we showed that the condition which must be satisfied so that the trajectory lies in a compact region of the Oxy plane is expressed by the inequalities:

$$1 - \varepsilon^2 > 0 \text{ or: } v_0 < \sqrt{\frac{2GM}{x_0}\left(1 - \frac{GM}{2x_0c^2}\right)} \text{ and}$$
$$\varepsilon > 0 \text{ or: } x_0 - \frac{GM}{c^2} > 0 \qquad (4.2)$$

In the Newtonian limit, from 4.2 we imply the condition $\frac{v_0{}^2}{2} - \frac{GM}{x_0} < 0,$ which means that to obtain a bounded trajectory, the energy of the moving particle must be negative.

The particle is moving around a spherical star of mass M and radius $R = x_0 / 4 = 0.25$ sim-units. In order to avoid the impact of the particle with the star, in the environment of each model, the initial velocity of the particle must be such that: $r_2 \geq x_0 / 4$ and $r_{2N} \geq x_0 / 4$ Under the conditions of the simulation, it holds that (see activity 7): $r_2 \geq r_{2N}$ for any value of v_0. Hence, from 3.1 we imply that the condition of non-impact with the star is:

$$v_0 \geq \sqrt{\frac{2GM}{5}} \qquad (4.3)$$

From 4.3 and 4.2 we infer that:

$$\sqrt{\frac{2GM}{5}} \leq v_0 < \sqrt{2GM\left(1 - \frac{GM}{2}\right)} \qquad (4.4)$$

In the environment of the simulation the mass M of the star is controlled by the user. Its max value is 40 sim-units and its min value 2 sim-units:

$2 \leq M \leq 40$ sim-units (4.5)

Activities

The quantities are measured in the system of units of the virtual environment of the simulation.

1) For M=20 calculate the initial velocity v_0 so that the trajectory of the moving particle be a circle of radius 1. Calculate the corresponding energy of the particle in both models. Repeat successively, for M=4 and M=40.
2) For M=30 calculate in the relativistic model the max value of v_0 so that the orbit of the particle be bounded. Do the same for M=4. Repeat the same calculations in the context of the Newtonian model.
3) For M=40 and $v_0 = 0.78$ measure the precession of the perihelion or the aphelion of the moving particle on the virtual environment of the simulation. Check your measurement by implementing the necessary theoretical calculations. Repeat the same activity for each of the following combinations:
   a) M=30 and $v_0 = 0.70$ b) M=4 and $v_0 = 0.20$ c) M=4 and $v_0 = 0.25$

4) For each case of activity 3, check if there is any visible precession in the context of the Newtonian model. Use the Newtonian virtual environment of the simulation.
5) For each case of activity 3, implement the following tasks:
    a. Calculate the aphelion r_a and the perihelion r_p of the orbit in the environment of the relativistic model. Check your results in the virtual environment of the simulation.
    b. Calculate the aphelion r_{aN} and the perihelion r_{pN} of the orbit in the environment of the Newtonian model. Check your results in the virtual environment of the simulation.
    c. The eccentricity of the path in both models is determined by the relation:
    $$E = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}$$
    Calculate the eccentricity for every case, in both models. Compare the results and formulate a conclusion.
6) Derive the conditions which should be satisfied between x_0, v_0 and M, so that the inequality $r_2 > x_0$ be true:
    a. in the Newtonian model,
    b. in the relativistic model.
    c. Check your results in the virtual environment of the simulation.
7) Let x_0, r_2 the extrema of the orbit in the relativistic model and x_0, r_{2N} the corresponding extrema in the Newtonian model, with the same initial velocity $(0, v_0, 0)$. Prove that if $\varepsilon^2 < 1$ then, it is true that $r_2 \geq r_{2N}$ for any value of v_0; the equality holds for the case of any circular orbit. Check the validity of this proposition in the environment of the simulation.

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