Elastic collisions between particles: The relativistic point of view

Kostas Papamichalis Dr. of Theoretical Physics

Synopsis
In this work, we compose the theoretical model of the elastic collision between two particles for the special case where the particles are merged into one point -their center of mass- and then after their interaction are emitted with new constant velocities [2]. During this process, each particle does not lose its identity; the (rest) mass of each particle remains invariant. This simple situation of emerging and emitting particles is a necessary restriction for the composition of the relativistic model: it implies that in the center-of-mass-frame there is a time-moment that the velocities of the particles are zero and the spatial parts of the four-forces by which the particles interact, are of the Newtonian action-reaction form. Hence, one can imply the four-momentum conservation during the particles' interaction, in any inertial-Cartesian reference frame.

Key-Concepts
Minkowski space – World lines of moving particles – Proper-time – World-time – Four-velocity – Four-momentum – Minkowski force – Inertial reference systems in Cartesian coordinates – The Lorentz transformations – The equations of motion in a Minkowski space – System of interacting particles – Center-of-Mass inertial system of reference – Four-momentum of a system – The four-momentum conservation for a system of particles

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Paragraph 1: The four-force and the Minkowski equations of motion

Consider two particles P1 and P2 with masses m_1, m_2 respectively, moving in the Minkowski space-time continuum *M*. In *M* we have defined an inertial, Cartesian coordinate system, named "Laboratory System (LS)". According to the LS, P2 has been placed at the point $(0,0,0,0)$ of *M* and P1 at the point $(0,-a,0,0)$, $a>0$

The clocks set at the spatial points of *M* are all synchronized, and they indicate the world time *t* [4,5] of M in the lab-system LS. The infinitesimal interval at any point of *M* is:

$$\Delta s = \left(g_{\mu\nu}\Delta x^{\mu}\Delta x^{\nu}\right)^{1/2}$$

…where:

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Hence:

$$\Delta s = \left(c^2\Delta t^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2\right)^{1/2} = \frac{c}{\gamma}\Delta t$$

$$\gamma(v) = \left(1 - \frac{\vec{v}\cdot\vec{v}}{c^2}\right)^{-1/2},\ \vec{v} = \frac{d\vec{r}}{dt} = \left(\dot{x}^1, \dot{x}^2, \dot{x}^3\right),\ v = \|\vec{v}\| = \sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2}$$

It is assumed that P1 and P2 interact only when their spatial distance, at the same time moment, is less than an infinitesimal positive quantity ε. Their world lines are solutions of the Minkowski equations of motion [2,5]:

Particle P1:

$$\frac{dP_1^{\mu}}{d\tau_1} = K_{12}{}^{\mu} \qquad (1.1a)$$

Particle P2:

$$\frac{dP_2^{\mu}}{d\tau_2} = K_{21}{}^{\mu} \qquad (1.1b)$$

With τ_1, τ_2 we symbolize the proper time of each particle, and with P_1, P_2 their four-momenta [1,2,3,4,5]. K_{12}, K_{21} are the Minkowski forces between the particles P1 and P2. The Greek indices take the values 0,1,2,3.

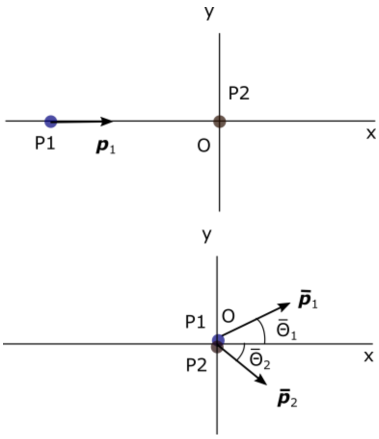

Figure 1: Configuration of the system in the context of our model.

For a particle with mass m, the four-momentum P is related to the four-velocity U: $P = mU$

The components of U are [2,5]:

$$U^{\mu} = \frac{dx^{\mu}}{d\tau},\ U^0 = \gamma c,\ U^j = \gamma\frac{dx^j}{d\tau} = \gamma v^j$$

The square of the norm of U is constant: $\langle U, U\rangle = g_{\mu\nu}U^m U^{\nu} = \gamma^2\left(c^2 - \vec{v}^2\right) = c^2$

Hence, it is implied that: $\left\langle \frac{dU}{d\tau}, U\right\rangle = 0$ and $\langle U, K\rangle = 0$, or:

$$cK^0 - \sum_{j=1}^{3} K^j v^j = 0 \qquad (1.2)$$

The four-momentum is also called "energy-momentum four-vector"; this is because its components can take the form:

$$P^0 = mc\gamma = \frac{E}{c},\ P^j = \gamma m v^j = \gamma p^j$$

...where $E = \dfrac{mc^2}{\sqrt{1 - \dfrac{\vec{v}\cdot\vec{v}}{c^2}}}$ is defined as the energy of the particle and $\vec{p} = m\vec{v}$ is the momentum according to the Newtonian definition.

By considering as free parameter the world time t, the equations of the motion take the form:

$$\frac{d}{dt}\left(m_1 c\gamma(v_1)\right) = \frac{1}{\gamma(v_1)}K_{(12)}{}^0,\ \frac{d}{dt}\left(m_1\gamma(v_1)v_1{}^j\right) = \frac{1}{\gamma(v_1)}K_{(12)}{}^j \qquad (1.3a)$$

$$\frac{d}{dt}\left(m_2 c\gamma(v_2)\right) = \frac{1}{\gamma(v_2)}K_{(21)}{}^0,\ \frac{d}{dt}\left(m_2\gamma(v_2)v_2{}^j\right) = \frac{1}{\gamma(v_2)}K_{(21)}{}^j \qquad (1.3b)$$

The initial four-velocities of P1 and P2 are, respectively:

$$U_1^0 = \gamma(v_0)c,\ U_1^0 = \gamma(v_0)cv_0,\ U_1^0 = 0,\ U_1^0 = 0$$

...where: $v_0 > 0$

$$U_2^0 = c,\ U_1^0 = 0,\ U_1^0 = 0,\ U_1^0 = 0$$

In the case of our model of the "elastic collision", we assume that the Minkowski force acting from each particle to the other is zero everywhere in the space-time continuum, except for an infinitesimal neighborhood around the event X_0 with coordinates $(ct_0, 0, 0, 0)$ (figure 1).

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Paragraph 2: Collision and emission of particles at a point of the Minkowski space

In the treatment of any many-particles problem like the collisions, a fundamental quantity is the total momentum of the particles. In Newtonian Mechanics, it is defined as the sum of the momentums of each particle. This summation is accomplished by the parallel transport of the momentum vectors at a specific point of the Euclidean space they are moving. This transportation

is legitimate because in Newtonian Mechanics the space is Euclidean and the time is an invariant quantity, irrelevant of space. *What about the Minkowski space-time continuum?*
The idea of the parallel transport of a vector is possible to get generalized for any space-time continuum determined by any metric tensor $\left[g_{\mu\nu}\right]$ (2,4,5). Anyway, for the case of the Minkowski space and an inertial-Cartesian coordinate system, it can be proved (2,4,5) that the coordinates of the parallel-transporting vectors do not change; we can define the total momentum of the particles as the four-vector P with coordinates:

$$P^{\mu} = P_1^{\mu} + P_2^{\mu} \text{ or: } P^0 = c\left(m_1\gamma_1 + m_2\gamma_2\right),\ P^j = m_1\gamma_1 v_1^{\ j} + m_2\gamma_2 v_2^{\ j}$$

…where: $\gamma_1 = \gamma(v_1), \gamma_2 = \gamma(v_2)$...

The Minkowski equations of motion are covariant, i.e., their form is the same in any inertial Cartesian coordinate system. Let us write 1.3 in the center-of-mass inertial system (CMS) of P1 and P2. This reference frame is defined by the conditions: a) the CM system is an inertial Cartesian coordinate system, b) the total spatial momentum of P1 and P2 is zero. Assume that the primed quantities refer to the CMS; then, we write:

$$P'^j = P_1'^j + P_2'^j = m_1\gamma_1' v_1'^{\ j} + m_2\gamma_2' v_2'^{\ j} = 0 \qquad (2.1)$$

Symbolize t' the world time of M in the CMS and conjecture that at the time t_0' both particles are at their center of mass and their velocities equal to zero: the particles "merge" into their center of mass and then "emerge" to different directions maintaining their masses m_1 and m_2.
The interaction of P1 and P2 takes place in an infinitesimal time-interval $\left(t_0' - \varepsilon, t_0' + \varepsilon\right)$, $\varepsilon \to 0^+$ In this interval the Minkowski forces acting on each particle have the form of a delta distribution around t_0' We can write:

$$\frac{d}{dt'}P_1'^j = \frac{1}{\gamma(v_1')}K_{12}'^{\ j}\delta\left(t' - t_0'\right),\ \frac{d}{dt'}P_2'^j = \frac{1}{\gamma(v_2')}K_{21}'^{\ j}\delta\left(t' - t_0'\right)$$

…where $K_{12}'^{\ j}$, $K_{21}'^{\ j}$ are constants.
By adding these relations, integrating from $t_0' - \varepsilon_1$ to $t_0' + \varepsilon_2$, $\varepsilon_1, \varepsilon_2 > 0$ and remembering that at t_0' the velocities of P1 and P2 equal to zero, we obtain the subsequent equations:

$$\left(P_1'^j + P_2'^j\right)_{t_0'+\varepsilon_1} - \left(P_1'^j + P_2'^j\right)_{t_0'-\varepsilon_2} = K_{12}'^{\ j} + K_{21}'^{\ j}$$

$$\lim_{\varepsilon_1 \to 0}\left(\left(P_1'^j + P_2'^j\right)_{t_0'+\varepsilon_1} - \left(P_1'^j + P_2'^j\right)_{t_0'-\varepsilon_2}\right) = 0 \Rightarrow K_{12}'^{\ j} + K_{21}'^{\ j} = 0$$

Hence, the total spatial momentum is the same before and after the interaction:

$$\left(P_1'^j + P_2'^j\right)_{t_0'+\varepsilon_1} = \left(P_1'^j + P_2'^j\right)_{t_0'-\varepsilon_2} = 0 \qquad (2.2a)$$

…and:

$$K_{12}'^{\ j} + K_{21}'^{\ j} = 0 \qquad (2.2b)$$

Assume that the double-primed quantities symbolize the situation of the system after the P1-P2 interaction, in the CMS. Then, from 2.2a we infer that:

$$m_1\gamma(v_1')v_1'^{\ j} + m_2\gamma(v_2')v_2'^{\ j} = m_1\gamma(v_1'')v_1''^{j} + m_2\gamma(v_2'')v_2''^{j} \qquad (2.2c)$$

From 1.3, 1.2, and 2.2b, it is implied that:

$$\frac{d}{dt'}\left(m_1 c\gamma(v_1') + m_2 c\gamma(v_2')\right) = \frac{1}{\gamma(v_1')}K_{12}'^{\ 0} + \frac{1}{\gamma(v_2')}K_{21}'^{\ 0} = \sum_{j=1}^{3}\left(\frac{c}{\gamma(v_1')}v_1'^{\ j} - \frac{c}{\gamma(v_2')}v_2'^{\ j}\right)K_{12}'^{\ j}\delta\left(t' - t_0'\right)$$

By integrating from $t_0' - \varepsilon_1$ to $t_0' + \varepsilon_2$, $\varepsilon_1, \varepsilon_2 > 0$ we obtain:

$$\left(m_1 c\gamma(v_1'') + m_2 c\gamma(v_2'')\right) - \left(m_1 c\gamma(v_1') + m_2 c\gamma(v_2')\right) = c\sum_{j=1}^{3}\left(\frac{1}{\gamma(v_1')}v_1'^{\ j} - \frac{1}{\gamma(v_2')}v_2'^{\ j}\right)K_{12}'^{\ j}\Bigg|_{t'=t_0'} = 0$$

We conclude that the total energy of the system is the same before and after the P1-P2 interaction:

$$m_1 c\gamma(v_1') + m_2 c\gamma(v_2') = m_1 c\gamma(v_1'') + m_2 c\gamma(v_2'')$$
$$\text{or: } \frac{E_1'}{c} + \frac{E_2'}{c} = \frac{E_1''}{c} + \frac{E_2''}{c} \qquad (2.3)$$

From 2.2c and 2.3 we deduce that in our model, the total four-momentum is invariant during the collision, for any choice of the inertial Cartesian system of reference.
P1 and P2 merge in their center of mass. In the CMS, this happens at the time t_0'. At this moment, both velocities of the particles equal to zero. The coordinates of the four-momentum for each particle are $(\gamma(0) = 1)$:
$P_1'(t_0') \to m_1(c,0,0,0)$ and $P_2'(t_0') \to m_2(c,0,0,0)$

Let t_0 be the corresponding moment in the lab-system (LS) that P1 and P2 merge (figures 2 and 3). Let $\left[\Lambda_\mu{}^\nu\right]$ be the Lorentz matrix relating the lab-system with the center of mass system. The spatial axes of the CMS and the LS are parallel, and CMS moves along the x^1 axis. Hence, the matrix $\left[\Lambda_\mu{}^\nu\right]$ has the form:

$$\left[\Lambda_\mu{}^\nu\right] = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.4)$$

The coordinates P^μ of the four-momentum for each particle in the lab-system at t_0, are calculated by the equations: $P^\mu(t_0) = P'^\nu(t_0')\Lambda_\nu{}^\mu$
We obtain:
$$\gamma(v_1)\big|_{t_0} = \gamma(v_2)\big|_{t_0} = \Lambda_0{}^0,\ \gamma(v_1)v_1{}^1\big|_{t_0} = \gamma(v_2)v_2{}^1\big|_{t_0} = c\Lambda_0{}^1 \Rightarrow v_1{}^1 = v_2{}^1,\ v_1{}^2 = v_2{}^2 = 0,\ v_1{}^3 = v_2{}^3 = 0$$
We conclude that in the lab-system, at the time t_0 when P1 and P2 merge, their velocities are equal.

The spatial part of the equations of motion, in whichever inertial Cartesian reference frame, are of the form:
$$\frac{d}{dt}\left(m_1\gamma_1\vec{v}_1\right) = \frac{1}{\gamma_1}\vec{K}_{12}\delta(t-t_0) \underset{def}{=} \vec{F}_{12},\ \frac{d}{dt}\left(m_2\gamma_2\vec{v}_2\right) = \frac{1}{\gamma_2}\vec{K}_{21}\delta(t-t_0) \underset{def}{=} \vec{F}_{21}$$
...where: $\vec{K}_{12} + \vec{K}_{21} = 0$ and: $\vec{v}_1(t_0) = \vec{v}_2(t_0)$
In the particular inertial frame, define the "angular momentum" of the system, by the usual in the Newtonian Mechanics, way:
$$\vec{L} = \vec{r}_1 \times \vec{P}_1 + \vec{r}_2 \times \vec{P}_2 = \vec{r}_1 \times \left(m_1\gamma_1\vec{v}_1\right) + \vec{r}_2 \times \left(m_2\gamma_2\vec{v}_2\right)$$
It holds:
$$\frac{d\vec{L}}{dt} = \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$
By integrating with time in the interval $\left[t_0 - \varepsilon_1, t_0 + \varepsilon_2\right]$, and taking into account that:
$\vec{r}_1(t_0) = \vec{r}_2(t_0)$, $\gamma(v_1)\big|_{t_0} = \gamma(v_2)\big|_{t_0}$
…we obtain the equation:
$$\vec{L}(t_0 + \varepsilon_1) - \vec{L}(t_0 - \varepsilon_2) = \left(\frac{1}{\gamma_1}\vec{r}_1 \times \vec{K}_{12} + \frac{1}{\gamma_2}\vec{r}_2 \times \vec{K}_{21}\right)_{t=t_0} = 0$$
We deduce that the "angular momentum" $\vec{L}$ of the system is conserved. If the initial positions and velocities of P1 and P2 lie on a certain plane, the motion of the particles takes place on that plane.

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Paragraph 3: The relativistic model of two-particle elastic collision

In our model, the lab-inertial system is symbolized (O,x). Point O is the origin of the spatial coordinates; its world-line is determined by the curve $(ct,0,0,0) \leftrightarrow O$ of the coordinate space R^4 The parameter t is the world time indicated by all the clocks placed at the spatial points of the Minkowski space M.
At t=0, particle P1 is at the point $P_0 \in M$ with coordinates $(0,-a,0,0)$, $a > 0$. P1 is moving toward O with constant velocity v_0; its world line for $t < t_0 = \dfrac{a}{v_0}$, is determined by the curve of the coordinate space with the analytic expression:

$$x_{P1}(t) = (ct, -a + v_0 t, 0, 0),\ t < t_0 \qquad (3.1a)$$

Particle P2 is at rest at the origin O of the lab-system. For $t < t_0$, its world line is determined by the curve of R^4:

$$x_{P2}(t) = (ct, 0, 0, 0),\ t < t_0 \qquad (3.1b)$$

When P1 reaches infinitely close to O, the particles interact, and they emerge with final velocities $\vec{\bar{v}}_1$, $\vec{\bar{v}}_2$ Their world lines in the coordinate space R^4 for $t > t_0 = \dfrac{a}{v_0}$ are of the form:

$$x_{P1}(t) = \left(ct, \bar{v}_{1x}\left(t - \frac{a}{v_0}\right), \bar{v}_{1y}\left(t - \frac{a}{v_0}\right), 0\right) \qquad (3.1c)$$

$$x_{P2}(t) = \left(ct, \bar{v}_{2x}\left(t - \frac{a}{v_0}\right), \bar{v}_{2y}\left(t - \frac{a}{v_0}\right), 0\right) \qquad (3.1d)$$

We must determine the velocities of P1 and P2 after their interaction. This is accomplished by implementing the following steps:
Step 1: Calculate the four-momentum of P1 and P2 in the CMS before their collision.
Step 2: Calculate the four-momentum of each particle in CMS after their collision and derive the analytic expression of each world line in CMS.
Step 3: Calculate the four-momentum of P1 and P2 in the LS after their collision. Derive the analytic expression of the particles' world lines in LS.

Step 1

Derive the Lorentz transformation connecting LS with CMS. Then, deduce the four-momentum of each particle in CMS before their collision

We use alternatively the symbolism: $P^1 \equiv P_x$, $P^2 \equiv P_y$, $P^3 \equiv P_z$, $v^1 \equiv v_x$... etc.

The four-momentum of each particle in LS before the collision is expressed by the following equations; see ref. 4 and ref. 5: Physics Problems (sch.gr)

For $t < t_0 = \dfrac{a}{v_0}$:

$$P_1 = e_0 \frac{E_1}{c} + e_1 P_{1x},\ E_1 = m_1 c^2 \gamma(v_0),\ P_{1x} = m_1 \gamma(v_0) v_0 \qquad (3.2a)$$

$$P_2 = e_0 \frac{E_2}{c},\ E_2 = m_2 c^2 \qquad (3.2b)$$

...where $\{e_0, e_1, e_2, e_3\}$ the basis vectors of the tangent vector space at any point of the Minkowski continuum M in the lab-system.
In CMS, the total spatial momentum of P1 and P2 is zero. For $t < t_0$:

$$P'_{1x} + P'_{2x} = 0 \qquad (3.3a)$$

The analytic expression of the Lorentz transformation connecting CMS and LS is written:

$x'^{\mu} = x^{\nu}\Lambda'_{\nu}{}^{\mu} + x_0{}^{\mu}$ (3.3b)

…where $\left[\Lambda'_{\nu}{}^{\mu}\right]$ is the inverse of the matrix $\left[\Lambda_{\nu}{}^{\mu}\right]$ (relation 2.5). We must determine the parameter θ and the constant quantities $x_0{}^{\mu}$ $\mu = 0,1,2,3$

Under the coordinate transformation with a Jacobian matrix given by 2.4, the basis vectors of the tangent vector-spaces of M, are transformed according to the relations [4,5]:

$e'_{\mu}\Delta x'^{\mu} = e'_{\mu}\Delta x^{\nu}\Lambda'_{\nu}{}^{\mu} = e_{\nu}\Delta x^{\nu} \Rightarrow e_{\nu} = e'_{\mu}\Lambda'_{\nu}{}^{\mu}$, $e'_{\mu} = e_{\nu}\Lambda_{\mu}{}^{\nu}$

Hence, from 3.2a, 3.2b, 3.3a, we obtain:

$$e_0\left(\frac{E_1}{c} - \Lambda_0{}^0\frac{E'_1}{c} - \Lambda_1{}^0 P'_{1x}\right) + e_1\left(P_{1x} - \Lambda_0{}^1\frac{E'_1}{c} - \Lambda_1{}^1 P'_{1x}\right) = 0$$

$$e_0\left(\frac{E_2}{c} - \Lambda_0{}^0\frac{E'_2}{c} - \Lambda_1{}^0 P'_{2x}\right) + e_1\left(-\Lambda_0{}^1\frac{E'_2}{c} - \Lambda_1{}^1 P'_{2x}\right) = 0$$

$$\tanh\theta = -c\frac{P_{1x}}{E_1 + E_2} \quad (3.4a)$$

$$\frac{E'_1}{c} = \frac{E_1}{c}\Lambda_1{}^1 - P_{1x}\Lambda_1{}^0,\ P'_{1x} = -\frac{E_1}{c}\Lambda_0{}^1 + P_{1x}\Lambda_0{}^0 \quad (3.4b)$$

$$\frac{E'_2}{c} = \frac{E_2}{c}\Lambda_0{}^0,\ P'_{2x} = -\frac{E_2}{c}\Lambda_0{}^1 \quad (3.4c)$$

Check that the length of the spatial momentum of each particle in CMS is:

$$\left\|\vec{P}'_1\right\| = \left\|\vec{P}'_2\right\| = \left|P'_{1x}\right| = \left|P'_{2x}\right| = \left|P_{1x}\right|\frac{E_2}{E_1 + E_2}\cosh\theta \quad (3.4d)$$

For the calculation of $x_0{}^{\mu}$ in 3.3b, we think as follows: Map the world lines of P1 and P2 in CMS, before the collision and demand that: a) when P1 is at the point $(0,-a,0,0)$ according to the LS, then the world time t' in CMS is zero, b) at the time $t_0 = \dfrac{a}{v_0}$, P1 is at the spatial origin of the LS, then P1 and P2 are both at the spatial origin of CMS. In the language of mathematics, we write:

From 3.3b:

$$ct' = ct\Lambda_0{}^0 - x\Lambda_1{}^0 + x_0{}^0,\ x' = -ct\Lambda_0{}^1 + x\Lambda_1{}^1 + x_0{}^1 \quad (3.5)$$

The world line of P1 in CMS:

$$ct'_1 = ct\Lambda_0{}^0 - (v_0 t - a)\Lambda_1{}^0 + x_0{}^0,\ x'_1 = -ct\Lambda_0{}^1 + (v_0 t - a)\Lambda_1{}^1 + x_0{}^1 \quad (3.6a)$$

The world line of P2 in CMS:

$$ct'_2 = ct\Lambda_0{}^0 + x_0{}^0,\ x'_2 = -ct\Lambda_0{}^1 + x_0{}^1 \quad (3.6b)$$

From 3.6a, for $t = 0, x_1 = -a$ we demand $t'_1 = 0$ It is implied that: $x_0{}^0 = -a\Lambda_1{}^0$

For $t = \dfrac{a}{v_0}$ we demand $x'_1 = 0$ Hence: $x_0{}^1 = \dfrac{ca}{v_0}\Lambda_0{}^1$

We infer that 3.5 takes the final form:

$$ct' = ct\Lambda_0{}^0 - \Lambda_1{}^0(x + a),\ x' = -c\Lambda_0{}^1\left(t - \frac{a}{v_0}\right) + x\Lambda_1{}^1 \quad (3.7a)$$

The non-zero matrix elements of $\left[\Lambda_{\mu}{}^{\nu}\right]$ are:

$\Lambda_0{}^0 = \Lambda_1{}^1 = \cosh\theta$, $\Lambda_0{}^1 = \Lambda_1{}^0 = -\sinh\theta$, $\Lambda_2{}^2 = \Lambda_3{}^3 = 1$

The inverse of 3.7a is determined by the transformation:

$$\begin{aligned} ct &= \Lambda_0{}^0\left(ct' + a\Lambda_1{}^0\right) + \Lambda_1{}^0\left(x' - \Lambda_0{}^1\frac{ac}{v_0}\right) \\ x &= \Lambda_0{}^1\left(ct' + a\Lambda_1{}^0\right) + \Lambda_1{}^1\left(x' - \Lambda_0{}^1\frac{ac}{v_0}\right) \end{aligned} \quad (3.7b)$$

The world lines of P1 and P2 for $t < \dfrac{a}{v_0}$, in the CMS, take the form:

$$ct_1' = ct\left(\Lambda_0^{\ 0} - \frac{v_0}{c}\Lambda_1^{\ 0}\right),\ x_1' = c\left(t - \frac{a}{v_0}\right)\left(-\Lambda_0^{\ 1} + \frac{v_0}{c}\Lambda_1^{\ 1}\right) \qquad (3.8a)$$

$$ct_2' = ct\Lambda_0^{\ 0} - a\Lambda_1^{\ 0},\ x_2' = -\Lambda_0^{\ 1} c\left(t - \frac{a}{v_0}\right) \qquad (3.8b)$$

In the analytic expressions 3.8a and b, the free parameter *t* is the world time being measured by the clocks fixed at the spatial-points of the lab-system. One can confirm that P1 and P2 arrive instantaneously at the origin of their center of mass system (CMS).

Step 2
Calculation of the four-momentum and the analytic expression of the world line for each particle in the CMS after the collision

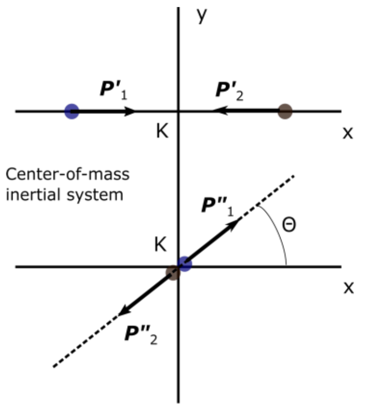

Figure 2: The P1-P2 collision in their center of mass frame.

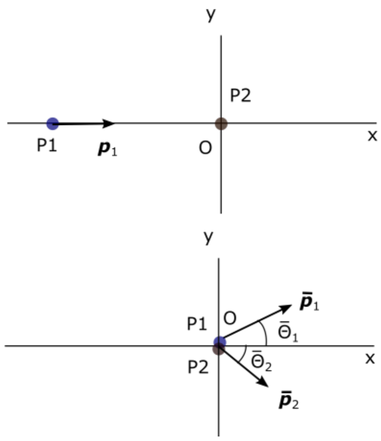

Figure 3: Motion of the particles after their collision, in the lab-system.

The double-primed quantities concern the situation of the system after the collision, in the CMS. According to paragraph 2, during the particles' collision the following statements are true:

a) The four-momentum of the system is conserved.
b) The total spatial momentum in the CMS equals zero.
c) The direction of each spatial momentum does change: the spatial momentums after the collision are placed on an axis forming an angle Θ with the x-axis. The value of Θ depends on the details of the interaction-mechanism between P1 and P2. In our application Θ is considered as an arbitrary parameter controlled by the user.

In the CMS, it holds:

$$P_1'' = e_0\frac{E_1''}{c} + e_1 P_{1x}'' + e_2 P_{1y}'',\ P_2'' = e_0\frac{E_2''}{c} + e_1 P_{2x}'' + e_2 P_{2y}''$$

$$E_1'' + E_2'' = E_1' + E_2',\ P_{1x}'' + P_{2x}'' = 0,\ P_{1y}'' + P_{2y}'' = 0$$

$$\frac{E_1''^{\,2}}{c^2} - \vec{P}_1''^{\,2} = m_1^{\,2}c^2,\ \frac{E_2''^{\,2}}{c^2} - \vec{P}_2''^{\,2} = m_2^{\,2}c^2$$

$$\frac{1}{c^2}\left(E_1'' - E_2''\right)\left(E_1'' + E_2''\right) = \left(m_1^{\,2} - m_2^{\,2}\right)c^2,\ E_1'' - E_2'' = \frac{\left(m_1^{\,2} - m_2^{\,2}\right)c^4}{E_1' + E_2'}$$

Hence:

$$E_1'' = \frac{1}{2}\left(E_1' + E_2' + \frac{\left(m_1^{\,2} - m_2^{\,2}\right)c^4}{E_1' + E_2'}\right),\ E_2'' = \frac{1}{2}\left(E_1' + E_2' - \frac{\left(m_1^{\,2} - m_2^{\,2}\right)c^4}{E_1' + E_2'}\right) \qquad (3.9a)$$

The length of each spatial momentum is calculated by the following equations:

$$\left\|\vec{P}_1''\right\| = \sqrt{\frac{E_1''^{\,2}}{c^2} - m_1^2c^2}\,,\ \left\|\vec{P}_2''\right\| = \sqrt{\frac{E_2''^{\,2}}{c^2} - m_2^2c^2} \tag{3.9b}$$

…and their coordinates (figure 2):

$$P_{1x}'' = -P_{2x}'' = \left\|\vec{P}_1''\right\|\cos\Theta\,,\ P_{1y}'' = -P_{2y}'' = \left\|\vec{P}_1''\right\|\sin\Theta \tag{3.9c}$$

Check that: $\left\|\vec{P}_1''\right\| = \left\|\vec{P}_2''\right\|$

We now derive the analytic expression of the particles' world lines before and after the collision, in the CMS, with free parameter the world time t measured in the LS. The following equations are true:

$$\gamma(v_1'') = \frac{E_1''}{m_1c^2}\,,\ v_{1x}'' = \frac{P_{1x}''}{m_1\,\gamma(v_1'')} = c^2\frac{P_{1x}''}{E_1''}\,,\ v_{1y}'' = \frac{P_{1y}''}{m_1\,\gamma(v_1'')} = c^2\frac{P_{1y}''}{E_1''}\,,\ v_{1z}'' = 0$$

$$\gamma(v_2'') = \frac{E_2''}{m_2c^2}\,,\ v_{2x}'' = \frac{P_{2x}''}{m_2\,\gamma(v_2'')} = c^2\frac{P_{2x}''}{E_2''}\,,\ v_{2y}'' = \frac{P_{2y}''}{m_2\,\gamma(v_2'')} = c^2\frac{P_{2y}''}{E_2''}\,,\ v_{2z}'' = 0$$

From 3.8a and b, and the previous equations, we deduce the analytic expressions:

$$ct_1' = ct\left(\Lambda_0^{\ 0} - \frac{v_0}{c}\Lambda_1^{\ 0}\right),\ x_1' = \begin{cases} c\left(-\Lambda_0^{\ 1} + \Lambda_1^{\ 1}\dfrac{v_0}{c}\right)\left(t - \dfrac{a}{v_0}\right) \text{ for } t < \dfrac{a}{v_0} \\ v_{1x}''\left(\Lambda_0^{\ 0} - \dfrac{v_0}{c}\Lambda_1^{\ 0}\right)\left(t - \dfrac{a}{v_0}\right) \text{ for } t > \dfrac{a}{v_0} \end{cases}$$

$$y_1' = \begin{cases} 0 \text{ for } t < \dfrac{a}{v_0} \\ v_{1y}''\left(\Lambda_0^{\ 0} - \dfrac{v_0}{c}\Lambda_1^{\ 0}\right)\left(t - \dfrac{a}{v_0}\right) \text{ for } t > \dfrac{a}{v_0} \end{cases},\ z_1' = 0 \tag{3.9d}$$

$$ct_2' = ct\Lambda_0^{\ 0} - a\Lambda_1^{\ 0}\,,\ x_2' = \begin{cases} -\Lambda_0^{\ 1}c\left(t - \dfrac{a}{v_0}\right) \text{ for } t < \dfrac{a}{v_0} \\ \Lambda_0^{\ 0}v_{2x}''\left(t - \dfrac{a}{v_0}\right) \text{ for } t > \dfrac{a}{v_0} \end{cases}$$

$$y_2' = \begin{cases} 0 \text{ for } t < \dfrac{a}{v_0} \\ \Lambda_0^{\ 0}v_{2y}''\left(t - \dfrac{a}{v_0}\right) \text{ for } t > \dfrac{a}{v_0} \end{cases},\ z_2' = 0 \tag{3.9e}$$

Step 3

Calculation of the four-momentum of P1 and P2 in the LS after their collision - Derivation of the analytic expression of the particles' world lines in the LS

The quantities with a bar concern the situation of the system after the P1-P2 collision, in the lab-system (LS). According to the previous steps, we obtain:

$$\bar{P}_1^{\ \mu} = P_1''^{\ \nu}\Lambda_\nu^{\ \mu}\,,\ \bar{P}_2^{\ \mu} = P_2''^{\ \nu}\Lambda_\nu^{\ \mu}$$

$$\frac{\bar{E}_1}{c} = \frac{E_1''}{c}\Lambda_0^{\ 0} + P_{1x}''\Lambda_1^{\ 0}\,,\ \bar{P}_{1x} = \frac{E_1''}{c}\Lambda_0^{\ 1} + P_{1x}''\Lambda_1^{\ 1}\,,\ \bar{P}_{1y} = P_{1y}'' \tag{3.10a}$$

$$\frac{\bar{E}_2}{c} = \frac{E_2''}{c}\Lambda_0^{\ 0} + P_{2x}''\Lambda_1^{\ 0}\,,\ \bar{P}_{2x} = \frac{E_2''}{c}\Lambda_0^{\ 1} + P_{2x}''\Lambda_1^{\ 1}\,,\ \bar{P}_{2y} = P_{2y}'' \tag{3.10b}$$

The spatial velocities of P1 and P2 in LS arise by the previous equations:

$$\gamma(\bar{v}_1) = \frac{\bar{E}_1}{m_1 c^2}, \; \bar{v}_1 = c\left(1 - \frac{1}{\left(\bar{E}_1 / m_1 c^2\right)^2}\right)^{1/2}$$

$$\bar{v}_{1x} = \frac{\bar{P}_{1x}}{m_1 \gamma(\bar{v}_1)} = c^2 \frac{\bar{P}_{1x}}{\bar{E}_1}, \; \bar{v}_{1y} = \frac{\bar{P}_{1y}}{m_1 \gamma(\bar{v}_1)} = c^2 \frac{\bar{P}_{1y}}{\bar{E}_1} \tag{3.11a}$$

$$\gamma(\bar{v}_2) = \frac{\bar{E}_2}{m_2 c^2}, \; \bar{v}_2 = c\left(1 - \frac{1}{\left(\bar{E}_2 / m_2 c^2\right)^2}\right)^{1/2}$$

$$\bar{v}_{2x} = \frac{\bar{P}_{2x}}{m_2 \gamma(\bar{v}_2)} = c^2 \frac{\bar{P}_{2x}}{\bar{E}_2}, \; \bar{v}_{2y} = \frac{\bar{P}_{2y}}{m_2 \gamma(\bar{v}_2)} = c^2 \frac{\bar{P}_{2y}}{\bar{E}_2} \tag{3.11b}$$

From 3.11a, b and 3.1a, b we derive the world lines of P1 and P2 in LS, before and after their collision (figure 3):

$$x_{P1}(t) = \begin{cases} \left(ct, -a + v_0 t, 0, 0\right), \; t < \dfrac{a}{v_0} \\ \left(ct, \bar{v}_{1x}\left(t - \dfrac{a}{v_0}\right), \bar{v}_{1y}\left(t - \dfrac{a}{v_0}\right), 0\right), \; t > \dfrac{a}{v_0} \end{cases} \tag{3.12a}$$

$$x_{P2}(t) = \begin{cases} \left(ct, 0, 0, 0\right), \; t < \dfrac{a}{v_0} \\ \left(ct, \bar{v}_{2x}\left(t - \dfrac{a}{v_0}\right), \bar{v}_{2y}\left(t - \dfrac{a}{v_0}\right), 0\right), \; t > \dfrac{a}{v_0} \end{cases} \tag{3.12b}$$

The angles $\bar{\Theta}_1, \bar{\Theta}_2, \bar{\Theta} = \bar{\Theta}_1 + \bar{\Theta}_2$ depicted in figure 3 are calculated by the equations:

$$\bar{\Theta}_1 = \cos^{-1}\left(\frac{\bar{v}_{1x}}{\bar{v}_1}\right), \; \bar{\Theta}_2 = \cos^{-1}\left(\frac{\bar{v}_{2x}}{\bar{v}_2}\right), \; \bar{\Theta} = \cos^{-1}\left(\frac{\vec{\bar{v}}_1 \cdot \vec{\bar{v}}_2}{\bar{v}_1 \bar{v}_2}\right), \; \bar{\Theta}_1, \bar{\Theta}_2, \bar{\Theta} \in \left(-\pi, \pi\right] \tag{3.13}$$

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