Plane oscillator: the relativistic and the Newtonian point of view

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Synopsis

In the context of the Newtonian Mechanics (NM), we consider a particle P moving in an inertial Cartesian reference frame (O,x,y,z), under the action of a force-field determined by the analytic expression: $\vec{F} = -k\vec{r}$; the vector $\vec{r}$ is the position vector of P and k a positive constant. This field defines a Newtonian oscillator with its equilibrium position at the origin O of the coordinate system. The trajectory of P lies on a plane determined by its initial position and velocity; the analytic expression of the trajectory is derived by the solution of a set of differential equations arising from Newton's 2nd law.

The treatment of the same problem in the context of Einstein's Special Relativity Theory (ESR) follows the same pattern, but there are two major differences: a) one has to deduce the analytic expression of the Minkowski four-force acting on P which in the Newtonian limit is compatible with the mentioned form and b) the trajectory of P arises as a solution of the generalization of the 2nd Newton's law in the four-dimensional Minkowski space-time continuum.

In the virtual environment of the application, the motion of the plane oscillator is simulated in the context of two models: the Newtonian and the relativistic. The comparison of the two viewpoints is accomplished for the same initial position and velocity of the oscillating particle.

It is worth mentioning that the path of the relativistic oscillator, although plane and localized, it is not a closed curve like the corresponding path in the Newtonian model.

Key-concepts

The Minkowski space – The metric tensor – Inertial reference systems in Cartesian coordinates – World line of a moving particle – Proper-time – Four-velocity – Synchronization of clocks in a Minkowski space – World time – Four-momentum of a moving particle – The four-force and the equations of motion in a Minkowski space

1. Synchronization of clocks in a Minkowski space [3,4] – The concept of the "world time"

In Newtonian mechanics, the free parameter of the trajectory's parametric equations is the absolute time t. In relativity, time is not absolute; it is one of the coordinates expressing the parametric equations of a world line. Nevertheless, it is possible to demonstrate that if M is a Minkowski space, and (O,x) an inertial, Cartesian coordinate system, all the clocks measuring the time coordinate at any space point of M can be synchronized. This "global" time is called "**world time**" (for details, about the way one can synchronize two clocks placed at two different space points of the space-time continuum, see paragraph 3 of reference 4).

In the virtual environment of the simulation, the chronometers are synchronized and measure the world time t. The world time is used as the free parameter in the analytic expressions of the particle's trajectories.

2. The Minkowski force and the Minkowski equations of motion [1,2,3,4]

The particles' equations of motion in a Minkowski space arise as a generalization of the 2nd Newton's law: the form of the equation of motion should be invariant under any coordinate transformation in M. We write:

$$\frac{DP}{D\tau} = K \text{ or: } \frac{DP^{\mu}}{D\tau} = K^{\mu} \tag{2.1}$$

K is a four-vector, called "the Minkowski four-force", τ the proper time of the moving particle, and P its four-momentum. Eq. 2.1 is called "the Minkowski equation of motion".

The Minkowski force is always orthogonal to the four-velocity U of the particle:

$$\left\langle \frac{DP}{D\tau}, U \right\rangle = \langle K, U \rangle = 0\,,\; K^0 c - \sum_{j=1}^{3} K^j v^j = 0 \tag{2.2}$$

The solution of the differential equations 2.1, under constraint 2.2 determines the world line of the moving particle (see paragraph 4 of reference 4). The analytic form of the world line is expressed by using as a free parameter the world time *t*. From 2.1 we obtain:

$$\gamma \frac{d}{dt}\left(m\gamma c\right) = K^0 \text{ , } \gamma \frac{d}{dt}\left(m\gamma v^j\right) = K^j \text{ , } \gamma = \left(1 - \frac{\vec{v}\cdot\vec{v}}{c^2}\right)^{-1/2} \quad (2.3)$$

The equation $\frac{d}{dt}\left(m\gamma v^j\right) = \frac{1}{\gamma}K^j$ converges to Newton's second law; for $\frac{v^2}{c^2} \to 0$ should be true that:

$\lim_{v/c\to 0}\left(\frac{1}{\gamma}K^j\right) = F_{(N)}{}^j$

How to create an acceptable form of the Minkowski force?

Assume that in the Newtonian model the force field acting on the moving particle comes from the potential energy $V(\vec{r})$:

$$F_j(\vec{r}) = -\partial_j V(\vec{r}) \quad (2.4)$$

We create the analytic expression of the Minkowski force so that the following two requirements are fulfilled: a) its components satisfy the constraints 2.2 and b) in the Newtonian limit, it converges to the equation 2.4.

In the 4th paragraph of the "Notes on the general principles of Relativistic Mechanics" (reference 4), it is demonstrated that an acceptable form of the quested Minkowski force is determined by the analytic expressions [(1)]:

$$K = e_\mu K^\mu \text{ , } K^\mu = -\frac{1}{c^2}V(x)\frac{dU^\mu}{d\tau} + \partial_\nu V(x)\left(g^{\mu\nu} - \frac{1}{c^2}U^\mu U^\nu\right) \quad (2.5)$$

From 2.5, we obtain the equations:

$$\begin{aligned} K^0 &= -\frac{1}{c^2}V(\vec{r})\frac{dU^0}{d\tau} + \partial_\nu V(\vec{r})\left(g^{0\nu} - \frac{1}{c^2}U^0U^\nu\right) = -\frac{1}{c}\gamma\left(V(\vec{r})\frac{d\gamma}{dt} + \gamma\partial_j V(\vec{r})v^j\right) = \\ &= -\frac{1}{c}\gamma\frac{d}{dt}\left(\gamma V(\vec{r})\right) \end{aligned} \quad (2.5a)$$

$$K^j = g^{jk}\partial_k V - \frac{1}{c^2}\gamma\frac{d}{dt}\left(\gamma V v^j\right) \quad (2.5b)$$

Hence, the equations of motion take the form:

$$\gamma\left(mc^2 + V\right) = mc^2 + w \ \left(w\text{=constant}\right) \quad (2.6a)$$

$$\gamma\frac{d}{dt}\left(\left(m + \frac{1}{c^2}V\right)\gamma v^j\right) = -\partial_j V \quad (2.6b)$$

It easily confirmed that 2.6a and b converge to the corresponding Newtonian equations for:

$\frac{\vec{v}^2}{c^2} \to 0$ and $\frac{V\left(\vec{r}\right)}{mc^2} \to 0$

3. Configuration of the differential equations of motion for the relativistic oscillator

The potential energy of the oscillator in the Newtonian model is given by the expression:

$V(r) = \frac{1}{2}kr^2$, $r = \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$

The Newtonian force $\vec{F}$ acting on the moving particle P is:

$F_j = -\partial_j V(r) = -\frac{dV(r)}{dr}\partial_j r = -kr\frac{x^j}{r} = -kx^j$, $j = 1,2,3$

The analytic expressions of the Cartesian coordinates for the corresponding Minkowski force are accomplished by 2.5a and b:

$$K^0 = -\gamma \frac{d}{dt}\left(\gamma \frac{k}{2c} r^2\right) \tag{3.1a}$$

$$K^j = -\partial_j V - \frac{1}{c^2}\gamma \frac{d}{dt}\left(\gamma V v^j\right) = -kx^j - \gamma \frac{d}{dt}\left(\frac{kr^2}{2c^2}\gamma v^j\right) \tag{3.1b}$$

The equations of motion arise from 2.6a and b:

$$\gamma\left(mc^2 + \frac{1}{2}kr^2\right) = mc^2 + w \tag{3.2a}$$

$$\gamma \frac{d}{dt}\left(\left(m + \frac{k}{2c^2}r^2\right)\gamma v^j\right) = -kx^j \tag{3.2b}$$

The quantity $w>0$ is interpreted as the "external" mechanical energy of the particle, which is necessary for the implementation of the oscillation.

Equations 3.2a and b concern the coordinate space R^4 ; by solving them we determine a curve c_P in R^4 which depicts the trajectory of P in the space-time continuum *M*.
From 3.2a and b we obtain the equations:

$$m\left(1 + \frac{w}{mc^2}\right)\frac{dv^j}{dt} = -\frac{k}{\gamma}x^j$$

$$\frac{d\vec{v}}{dt} = -\frac{k}{m\left(1 + \frac{w}{mc^2}\right)}\frac{\vec{r}}{\gamma} \tag{3.2c}$$

From the form of 3.2c, we conclude that the quantity $\vec{L} = \vec{r} \times (m\vec{v})$ is invariant along the curve c_P :

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times (m\vec{v}) + \vec{r} \times \frac{d}{dt}(m\vec{v}) = m\vec{v} \times \vec{v} - \frac{k}{\left(1 + \frac{w}{mc^2}\right)\gamma}\vec{r} \times \vec{r} = 0$$

$\vec{L}$ could be viewed as the "angular momentum" of the particle P defined in the coordinate space R^4
Hence, the curve c_P lies on a fixed plane determined by the initial position and velocity of the particle P. In the simulation, we have assumed that the initial conditions are: $\vec{r}(0) = r_0\hat{x}$, $\vec{v}(0) = v_0\hat{y}$. The angular momentum is the constant vector:

$\vec{L} = (r_0\hat{x}) \times (mv_0\hat{y}) = mr_0v_0\hat{z}$

The plane of the curve c_P is perpendicular to $\hat{z}$ and coincides with the (Ox^1x^2) plane.

To get the formalism easier, set: $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$ and $v^1 \equiv v_x, v^2 \equiv v_y, v^3 \equiv v_z$

Given that c_P lies on the (Oxy) plane, for every *t* it holds $z = 0$ and $v_z = 0$, and the equations of the motion are being simplified according to the relations:

$$\gamma\left(1 + \frac{k}{2mc^2}r^2\right) = 1 + \frac{w}{mc^2} \tag{3.3a}$$

$$\frac{dv_x}{dt} = -\frac{k}{m\left(1 + \frac{w}{mc^2}\right)}\frac{x}{\gamma} \tag{3.3b}$$

$$\frac{dv_y}{dt} = -\frac{k}{m\left(1 + \frac{w}{mc^2}\right)}\frac{y}{\gamma} \tag{3.3c}$$

We define the following parameters, which make the form of the differential equations as easier as possible, and will be proved useful in the composition of the program of the simulation:

a) The magnitude $b = \frac{v_0}{c}$ of the initial velocity

b) The ratio h_p of the initial potential energy $w_p = V\left(r_0\right)$ over the rest energy mc^2:

$h_p = \frac{V(r_0)}{mc^2} = \frac{kr_0^2}{2mc^2}$

The parameter h_p takes values in the interval $\left(0,1\right)$

The other parameters appearing in equations 3.3 are determined as functions of h_p and b:

$$h \underset{def}{=} \frac{w}{mc^2} = \frac{h_p + 1}{\sqrt{1-b^2}} - 1 = \left(h_p + 1\right)\gamma(b) - 1 \tag{3.4a}$$

$$k = h_p \frac{2mc^2}{r_0^2} \tag{3.4b}$$

Equations 3.3 are expressed in the equivalent forms:

$$1 + h_p \frac{r^2}{r_0^2} = \left(1 + h\right)\gamma^{-1}(v),\ v = \sqrt{{v_x}^2 + {v_y}^2} \tag{3.5a}$$

$$\frac{dv_x}{dt} = -\frac{2c^2 h_p}{\left(1+h\right){r_0}^2}\gamma^{-1}(v)x \tag{3.5b}$$

$$\frac{dv_y}{dt} = -\frac{2c^2 h_p}{\left(1+h\right){r_0}^2}\gamma^{-1}(v)y \tag{3.5c}$$

4. Description of the Newtonian oscillator with the same initial conditions

The Newtonian equations of motion arise from 4.4a-c, for $\frac{v^2}{c^2} << 1$ and $\frac{w}{mc^2} << 1$ We come to the following result:

$$\frac{k}{2}r^2 + \frac{1}{2}mv^2 = w \tag{4.1a}$$

$$\frac{dv_x}{dt} = -\frac{k}{m}x \tag{4.1b}$$

$$\frac{dv_y}{dt} = -\frac{k}{m}y \tag{4.1c}$$

The solutions of 4.1a to c, are:

$$x_N = r_0 \cos\left(f_{rad}t\right),\ y_N = r_1 \sin\left(f_{rad}t\right) \tag{4.2a}$$

$$v_{Nx} = -r_0 f_{rad} \sin\left(f_{rad}t\right),\ v_{Ny} = r_1 f_{rad} \cos\left(f_{rad}t\right) \tag{4.2b}$$

$$f_{rad} = \sqrt{\frac{k}{m}} \tag{4.2c}$$

The parameters *w, k*, f_{rad} and the semi-axis r_1 can be expressed with respect to h_p and b:

$$w = \frac{1}{2}mv_0^2 + \frac{1}{2}kr_0^2,\ h = \frac{b^2}{2} + h_p \tag{4.3}$$

$$\frac{1}{2}kr_0^2 = h_p mc^2, k = \frac{2h_p}{r_0^2}mc^2 \tag{4.4}$$

$$f_{rad} = \sqrt{\frac{k}{m}} = \sqrt{2c^2 \frac{h_p}{r_0^2}} = \frac{c}{r_0}\left(2h_p\right)^{1/2} \tag{4.5}$$

$$r_1 f_{rad} = v_0,\ r_1 = \frac{v_0}{f_{rad}} = r_0 \frac{b}{\sqrt{2h_p}} \tag{4.6}$$

From 4.2a we derive the equation of the elliptic path for the Newtonian plane oscillator:

$$\frac{x_N^2}{r_0^2}+\frac{y_N^2}{r_1^2}=1 \tag{4.7}$$

The Newtonian oscillator in polar coordinates

In polar coordinates the following equations are valid:

$x=r\cos\theta$, $y=r\sin\theta$

$v_x=\dot r\cos\theta-r\dot\theta\sin\theta$, $v_y=\dot r\sin\theta+r\dot\theta\cos\theta$

$v^2=\dot r^2+r^2\dot\theta^2$, $\dot r=\dfrac{dr}{dt}$, $\dot\theta=\dfrac{d\theta}{dt}$

$\tan\theta_N=\dfrac{r_0}{r_1}\tan\left(f_{rad}t\right)$

$r_N=\left(r_0^2\cos^2\left(f_{rad}t\right)+r_1^2\sin^2\left(f_{rad}t\right)\right)^{1/2}$

$r_N=\dfrac{r_1}{\sqrt{1-\left(1-\dfrac{r_1^2}{r_0^2}\right)\cos^2\theta_N}}$

For $t=\dfrac{\pi}{2f_{rad}}$, the distance from O is $r_N=r_1$ and the corresponding angular displacement from the initial position is:

$\theta_N=\tan^{-1}\left(\dfrac{r_0}{r_1}\tan\left(f_{rad}\dfrac{\pi}{2f_{rad}}\right)\right)=\dfrac{\pi}{2}$

We derive the differential equations of motion by using the two invariant quantities: the angular-momentum and the mechanical energy:

$w=\dfrac{1}{2}m\left(\dot r^2+r^2\dot\theta^2\right)+\dfrac{1}{2}kr^2$

$L=mr_0v_0=mr^2\dot\theta$

Define $q\underset{def}{=}r/r_0$, then from the previous equations we obtain:

$$\dot q=\frac{c}{r_0}\sqrt{\left(q^2-1\right)\left(\frac{b^2}{q^2}-2h_p\right)} \tag{4.8a}$$

$$\frac{d\theta}{dt}=\frac{v_0}{r_0}\frac{1}{q^2}=\frac{c}{r_0}\frac{b}{q^2} \tag{4.8b}$$

$$\frac{dq}{d\theta}=\frac{1}{b}q^2\sqrt{\left(q^2-1\right)\left(\frac{b^2}{q^2}-2h_p\right)} \tag{4.8c}$$

According to the explicit form of the solution we have already derived in the Cartesian coordinate system, we imply that:

$$\theta(r_1)-\theta(r_0)=b\int_{q=1}^{q=r_1/r_0}dq\,\frac{1}{q^2}\left(\left(q^2-1\right)\left(\frac{b^2}{q^2}-2h_p\right)\right)^{-1/2}=\frac{\pi}{2} \tag{4.8d}$$

5. The relativistic oscillator's equations of motion in polar coordinates – Extrema of the path – Angular displacement of the extreme points of the path

In the relativistic model, we follow the same steps as in the Newtonian model. We result that:

Energy conservation (see 3.2a):

$$\left(1+h_p\frac{r^2}{r_0{}^2}\right)^2=\left(1+h\right)^2\left(1-\frac{1}{c^2}\left(\dot{r}^2+r^2\dot{\theta}^2\right)\right) \tag{5.1}$$

Angular-momentum conservation:
$L=mr_0v_0=mr^2\dot{\theta}$

$$\frac{d\theta}{dt}=\frac{r_0v_0}{r^2} \tag{5.2}$$

From 5.1 and 2 we obtain the differential equation:

$$\left(1+h_p\frac{r^2}{r_0^2}\right)^2=\left(1+h\right)^2\left(1-\frac{1}{c^2}\left(\dot{r}^2+\frac{r_0^2v_0^2}{r^2}\right)\right) \tag{5.3a}$$

Define: $q \underset{def}{=} \frac{r}{r_0}$ Then, 5.3a takes the form:

$$\dot{q}=\frac{c}{r_0}F_{rel}\left(q\right) \text{ where: } F_{rel}\left(q\right)\underset{def}{=}\sqrt{1-\frac{\left(1+h_pq^2\right)^2}{\left(1+h_p\right)^2}\left(1-b^2\right)-\frac{b^2}{q^2}} \tag{5.3b}$$

$$\frac{dq}{d\theta}=\frac{1}{b}q^2F_{rel}\left(q\right) \tag{5.3c}$$

The extreme values of r satisfy the condition $\dot{q}=0$; hence from 5.3b we imply that the extreme values of r are roots of the equation:

$$F_{rel}\left(q\right)=0 \text{ or: } 1-\frac{\left(1+h_pq^2\right)^2}{\left(1+h_p\right)^2}\left(1-b^2\right)-\frac{b^2}{q^2}=0 \tag{5.4}$$

It is noticed that:

$$\left.\frac{dr}{dt}\right|_{t=0}=\frac{1}{r}\left(xv_x+yv_y\right)\Big|_{t=0}=\frac{1}{r_0}\left(0+0\right)=0$$

…meaning that $r=r_0$ or: $q=1$ must be a root of 5.4, which is immediately confirmed.
The roots of 5.4 that are different from 1 are roots of the equation:

$$b^2-2h_pq^2+b^2h_p{}^2\left(q^4+q^2+1\right)+\left(2b^2h_p-h_p{}^2q^2\right)\left(q^2+1\right)=0 \tag{5.5}$$

We approximate the left part of 5.5 by keeping terms up to the first order of h_p and b^2
We result in the expression:

$$q_1^2\approx\frac{b^2}{2h_p} \tag{5.6}$$

which agrees with the length r_1 of the second semi-axis obtained in the context of the Newtonian model (relation 4.6).

The angular displacement of the oscillator between two positions corresponding to successive extrema of r is calculated by the equation:

$$\Theta=b\int_{q=1}^{q=r_1/r_0}dq\,\frac{1}{q^2}F_{rel}\left(q,h,h_p\right)^{-1} \tag{5.7}$$

In the virtual environment of the simulation, the user can estimate Θ by making measurements on the virtual path, or the real-time graphs, and compare the resulted values.

4. Description of the virtual environment – Units, Input, and Output-Tools

In the virtual environment of the simulation, the physical quantities are measured in atomic units (au). It is given that: c=1au, m=2000au and r_0=10au.

The particle's initial position and velocity are determined by the conditions:

$x(0) = r_0 = 10\text{au}, y(0) = 0, v_x(0) = 0, v_y(0) = v_0$

The user controls the parameters:

a) The initial velocity $v_0 \in [0.1c, 0.99c]$

b) The ratio h_p of the initial potential energy $w_p = V(r_0)$ over the rest energy mc^2:

$$h_p = \frac{V(r_0)}{mc^2} = \frac{kr_0^2}{2mc^2}$$

The parameter h_p takes values in the interval $[0.02, 2]$

Activities in the virtual environment of the simulation

Activity 1

Prove that if we know the initial values *h* and h_p then, the magnitude v_0 of the initial velocity should have the value: $v_0 = c\sqrt{1 - \left(\frac{1+h_p}{1+h}\right)^2}$

Set $h = 2$ and $h_p = 0.8$ and run the simulation.

A) By using the graphs and the available tools, calculate and write down the extreme values of the length *r* of the position vector $\vec{r}$ for each oscillator.

B) Calculate the time lapse between two successive extrema of the length r for each oscillator.

C) Calculate the period of the Newtonian oscillator.

D) Calculate the angle formed by the position vectors of the relativistic oscillator corresponding to the first and third points of the path with the maximum distance from O.

Activity 2

Prove that a sufficient condition for the Newtonian oscillator to draw a circular path with initial velocity $(0, v_0, 0)$ and position $(r_0, 0, 0)$ is given by the relation:

$$h_p = \frac{v_0^2}{2c^2}$$

Check this condition in the virtual environment of the simulation for a) $v_0 = 0.2\text{au}$ and b) $v_0 = 0.8\text{au}$

For case (a), is the shape of the relativistic oscillator's path anticipated? Explain.

For case (b): Calculate the angular displacement of the greater extremum of the distance r from the origin O (see activity 1D).

Activity 3

Set $v_0 = 0.99\text{au}, h_p = 0.8\text{au}$. By using the graphs and the available measurement tools, calculate the angular displacement of the max extremum of each oscillator (see activity 1D).

Repeat for: $v_0 = 0.6\text{au}$, $h_p = 0.2\text{au}$ and: $v_0 = 0.6\text{au}$, $h_p = 2\text{au}$

Activity 4

Prove that the condition for the relativistic oscillator to follow a circular path with radius r_0 and velocity of magnitude v_0 is given by the relation (hint: use relation 5.5):

$$h_p = \frac{b^2}{2 - 3b^2} \quad \text{where: } b = \frac{v_0}{c}$$

Check the previous condition in the virtual environment of the simulation for $v_0 = 0.6\text{au}$, $h_p = 0.39\text{au}$

In that case, calculate the length of the small and the great semi-axis of the Newtonian oscillator, its period, and the constant *k* of the restoring force.

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