The constrained motion of a particle along a curve: Newtonian and Relativistic models

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Simulation:
http://users.sch.gr/kostaspapamichalis/ejss_model_constrMotion/index.html

Abstract

In this work, we study, simulate and compare the constrained motion of a particle within two theoretical models: a relativistic and a Newtonian. The main objective is to get the user acquainted with the differences emerging in the treatment of the same mechanical system within the frames of Newtonian and the relativistic point of view. In the virtual environment of the simulation, the user can change the initial state of the moving particle and the gravity field within which it moves; so he can notice the resulting variations in the motion of the system.
To achieve the objectives we have set, the subsequent issues have been implemented:

a) In the context of Newtonian Mechanics, determine and study a mechanical system consisted of a particle moving on a surface or curve of the three-dimensional Euclidean space, in the presence of a homogeneous gravitational field. In particular, we have considered the motion of a bead along a vertical circular wire. The cause for that choice is that in the Newtonian context, the motion of the bead is identical with the motion of a particle tied at the free end of a simple pendulum; although the simple pendulum is not possible to be realized in the context of the relativity theory, the bead moving along a predefined path does not present insuperable problems. Hence we are in the position to compare the predictions obtained by the Newtonian and the relativistic model for the evolution of a well-known mechanical system.
b) Study the same mechanical system in the context of the General Theory of Relativity.
c) Simulate the motion of the particle according to the Newtonian and the relativistic model and compare their predictions.

The simulation has been compiled in JavaScript language on the Easy JavaScript Simulations platform.

1) Determination of the mechanical system - The Newtonian model

Let us consider an inertial reference frame in Cartesian coordinates $Ox^1x^2x^3$ and a homogeneous gravitational field $\boldsymbol{g}$ directed along the negative semi-axes x^2. A bead B of mass m is constrained to be moving without friction, along a given curve c lying on the plane Ox^1x^2 (figure 1). Assume that the analytic expression of curve c in polar coordinates is:

$r = f(\theta)$ (1.1)

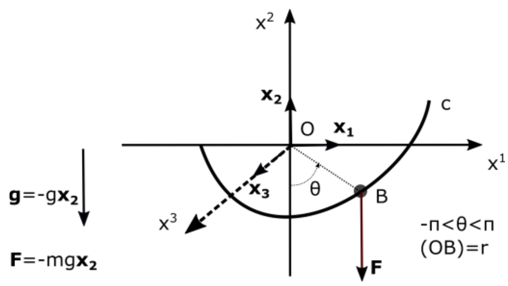


Figure 1

The form of the function $f(\theta)$ is known.

From Newton's 2nd low we obtain the equations:

$$\ddot{\vec{x}} = \frac{1}{m}\left(\vec{F}_c + \vec{F}\right) \quad (1.2)$$

We have symbolized $\vec{x} = \left(x^1, x^2, 0\right)$ the position vector of the bead, in Cartesian coordinates. The external force on B is symbolized by $\vec{F}$ and $\vec{F}_c$ is the force acted on it by the constraint. The constraint's force is perpendicular to the infinitesimal displacement $\Delta\vec{x}$ of the bead at any position of B on curve c. Hence:

$$\vec{F}_c \cdot \Delta\vec{x} = 0, \text{ or: } \vec{F}_c \cdot \dot{\vec{x}}\Delta t = 0, \ \vec{F}_c \cdot \dot{\vec{x}} = 0 \quad (1.3)$$

From 1.2 and 1.3 we eliminate $\vec{F}_c$ and we obtain the equation:

$$\ddot{\vec{x}} \cdot \Delta\vec{x} = \frac{1}{m}\vec{F} \cdot \Delta\vec{x} \quad (1.4a)$$

The external force $\vec{F}$ comes from the gravitational potential:

$$V\left(\vec{x}\right) = gx^2 = -gr(\theta)\cos\theta \quad (1.4b)$$

$\vec{F} = -m\left(\hat{x}_1\partial_1 V + \hat{x}_1\partial_2 V\right)$

Equation 1.4a takes the form:

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\vec{x}}^2 + V\left(\vec{x}\right)\right) = 0 \quad (1.5a)$$

Or:

$$\frac{1}{2}\dot{\vec{x}}^2 + V\left(\vec{x}\right) = \frac{E}{m} \qquad (1.5b)$$

E is the constant energy of the bead along its trajectory.

We combine the constraint-equation 1.1 with 1.5b; in polar coordinates, we derive the differential equation of the bead-motion along c:

$$\frac{1}{2}m\dot{\theta}^2\left(f'^2 + f^2\right) + V\left(r,\theta\right) = \frac{E}{m} \qquad (1.5c)$$

$$f'(\theta) \underset{def}{=} \frac{df(\theta)}{d\theta}$$

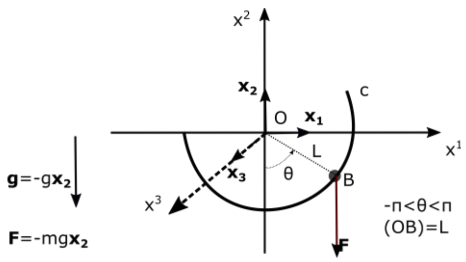


Figure 2

Assume that curve c is circular with radius L (figure 2). Then from equation 1.5c, we imply the following:

$$\frac{1}{2}L^2\dot{\theta}^2 - gL\cos\theta = \frac{E}{m} \qquad (1.6a)$$

Or:

$$\dot{\theta}\left(\ddot{\theta} + \frac{g}{L}\sin\theta\right) = 0 \qquad (1.6b)$$

From 1.6b, for the case that the bead is not at rest in an equilibrium state, we result in the differential equation of motion:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \qquad (1.6c)$$

Equation 1.6c is identical to a simple pendulum equation of motion, with rod-length equal to L.

In the next paragraph, we describe the same mechanical system in the context of General relativity. We simulate each model and compare its predictions.

2) The relativistic model (1,2,3,7,12)

The equation of motion of a particle in a space-time continuum with no constraints

In the General Relativity context, a particle P of mass m moving in a gravitational field is described as a free particle (9) moving in a space-time continuum M_4. The metric tensor of M_4 is determined by the specific gravitational field in the chosen coordinate system. For the case of a homogeneous gravitational field with potential $V\left(\vec{x}\right)$, $\vec{x} = \left(x^1, x^2, x^3\right)$ corresponding to the Newtonian model described in paragraph 1, in Cartesian coordinates the metric tensor is determined by the matrix (1,2,12):

$$\left[g_{\mu\nu}\right] = \begin{pmatrix} 1 + \frac{2V}{c^2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (2.1)$$

The analytic expression of the scalar potential V is given by 1.4b.

It is necessary to notice that the space-time continuum M_4 equipped with the metric tensor 2.1 has the following properties:

a) Each point X of the space-time continuum M_4 is uniquely determined by a set of Cartesian coordinates(2,4,10,12): $x = \left(x^0, x^1, x^2, x^3\right) \in R^4$ I.e. the space-time continuum M_4 is determined by a one-to-one differentiable function(8) of R^4 to R^4:

$R^4 \ni x \to X = \Phi(x) \in R^4$, $\Phi\left(R^4\right) \underset{def}{=} M_4 \subseteq R^4$

The x^0 is called "time-coordinate" and the other three "spatial-coordinates".

At each point $X \in M_4$ is defined a tangent vector space $T_X M_4$ spanned by the linearly independent basis-vectors:

$e_\mu(x) = \partial_\mu \Phi(x)$, $\mu = 0,1,2,3$

The metric tensor defines on every tangent space $T_x M_4$ an inner product according to the equalities:

$\langle e_\mu(x), e_\nu(x) \rangle = g_{\mu\nu}(x)$, $\mu, \nu = 0,1,2,3$

The "interval" Δs between two infinitesimally close points X and $X + \Delta X$ of M_4 is calculated by the relationships:

$$X = \Phi(x), X + \Delta X = \Phi(x + \Delta x) \in M_4$$

$$\Delta X = \partial_\mu \Phi(x) \Delta x^\mu = e_\mu(x) \Delta x^\mu$$

$$\Delta s^2 = \langle \Delta X, \Delta X \rangle = \langle e_\mu(x), e_\nu(x) \rangle \Delta x^\mu \Delta x^\nu$$

b) The gravitational field determined by the metric tensor 2.1 is **static**(1). Hence we can synchronize the clocks set at every spatial point of the space-time continuum.

c) The spatial geometry (or: every "rest-space"(2)) of the space-time continuum is Euclidean(1,2). That means the snapshot of M_4 at any instant of time, is a three-dimensional Euclidean space(1,2,12).

According to property b, we can study the motion of any particle in M_4 by choosing as "free parameter" the "**world time**"(1) t measured by an observer placed at the spatial origin O of the Cartesian coordinate system.

The world line of a particle P moving in M_4 is a geodetic of M_4, i.e. a solution of the differential equation(1,2,10,12):

$$D_{\Delta X} U(s) = 0 \qquad \text{where: } \Delta X = U(s) \Delta s \qquad (2.2a)$$

Parameter s is the interval along the geodetic.

With the symbol: $D_{\Delta X} U(s)$ we symbolize the covariant differential of the particle's four-velocity $U(s)$ along the infinitesimal displacement ΔX on its world-line(1,2,4,10,12).

The four-velocity and the infinitesimal displacement are vector fields defined on the "tangent bundle"(2,12) of the space-time continuum M_4. The following identities are satisfied:

$$U = \frac{dX}{ds},\ U = e_\mu(x) u^\mu,\ u^\mu = \frac{dx^\mu}{ds}$$

$$\Delta s = c\Delta t \sqrt{g_{00} - \frac{\vec{v} \cdot \vec{v}}{c^2}} \underset{def}{=} \frac{1}{\gamma} c\Delta t$$

Where: $g_{00} = 1 + \frac{2V}{c^2}$, $\vec{v} = \frac{d}{dt}(x^1, x^2, x^3)$, $\gamma = \left(g_{00} - \frac{\vec{v} \cdot \vec{v}}{c^2}\right)^{-1/2}$, $\vec{v} \cdot \vec{v} = (v^1)^2 + (v^2)^2 + (v^3)^2$

Covariant differentiation is determined through the concepts of the "parallel displacement" and the "connection" defined on the tangent bundle of the space-time continuum M_4(1,2,4,10,12). The connection we usually consider is compatible with the metric tensor of M_4; it is determined by the "Christoffel symbols" which are functions of the metric tensor's derivatives(1,2,12). In particular, one can verify the following identities:

$$D_{\Delta X}{}^\mu U = \Delta u^\mu + \Gamma^\mu{}_{\nu\kappa} u^\nu \Delta x^\kappa \qquad (2.2b)$$

The Christoffel $\Gamma^\mu{}_{\nu\kappa}$ symbols are calculated by the equations:

$$\Gamma_{\mu\nu\rho} = \frac{1}{2}\left(-\partial_\mu g_{\nu\rho} + \partial_\rho g_{\mu\nu} + \partial_\nu g_{\rho\mu}\right) \qquad \Gamma^\mu{}_{\nu\rho} = g^{\mu\kappa} \Gamma_{\kappa\nu\rho} \qquad (2.2c)$$

We combine relations 2.2a-c and we result in the differential equations of motion for a free particle in M_4:

$$\Delta u^\mu + \Gamma^\mu{}_{\nu\kappa} u^\nu \Delta x^\kappa = 0$$

$$\frac{du^\mu}{ds} + \Gamma^\mu{}_{\nu\kappa} u^\nu u^\kappa = 0 \qquad (2.2d)$$

Or:

$$\frac{d}{dt}\left(\gamma \frac{dx^\mu}{dt}\right) + \gamma \Gamma^\mu{}_{\nu\kappa} \frac{dx^\nu}{dt} \frac{dx^\kappa}{dt} = 0 \qquad (2.2e)$$

Where:

$$ds = \frac{cdt}{\gamma}, \ \gamma = \left(g_{00} - \frac{\vec{v}\cdot\vec{v}}{c^2}\right)^{-1/2}$$

The equation of motion of a particle sliding along a circular curve

Consider a particle P of mass m in the space-time continuum M_4 which is constrained to slide along a circular curve, like the bead we have studied in paragraph 1. M_4 is endowed with the metric tensor given by 2.1 and, as we have already mentioned, its spatial geometry is Euclidean. That means, a circular curve of radius L centered at the spatial origin O, lying on the spatial plane Ox^1x^2 of M_4, is determined by the analytic expression:

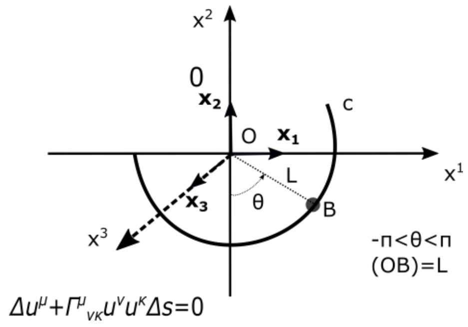


Figure 3

$$\left(x^1\right)^2 + \left(x^2\right)^2 - L^2 = 0 \qquad (2.3)$$
$$x^3 = 0$$

Hence, our particle P is confined to move in a sub-manifold P_L of the space-time continuum M_4 determined by 2.3. In this sub-manifold, P is moving as a free particle. That is, P moves along some geodetic of sub-manifold P_L. To derive the motion-differential equations, we have to find the induced metric tensor in P_L and the corresponding connection.

According to 2.3, we'll make our job easier if we jump to polar coordinates in M_4, by considering the following transformation (figure 3):

$$x^0 = x^0\left(=ct\right), \ x^1 = r\sin\theta, \ x^2 = -r\cos\theta, \ x^3 = z \qquad (2.4)$$

The coordinates of the tangent vectors transform according to the linear transformation:

$$\Delta x^0 = c\Delta t \qquad (2.4a)$$
$$\Delta x^1 = r\Delta\theta\cos\theta + \Delta r\sin\theta \qquad (2.4b)$$
$$\Delta x^2 = r\Delta\theta\sin\theta - \Delta r\cos\theta \qquad (2.4c)$$
$$\Delta x^3 = \Delta z \qquad (2.4d)$$

Or:

$$\left(\Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3\right) = \left(\Delta x^0, \Delta\theta, \Delta r, \Delta z\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r\cos\theta & r\sin\theta & 0 \\ 0 & \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Equations 2.3 that determine the sub-manifold P_L, in polar coordinates are transformed to the next:

$$r = L, \ z = 0 \qquad (2.5)$$

In polar coordinates, the interval Δs of the space continuum M_4 is expressed by the identity:

$$\Delta s^2 = g_{00}\left(\Delta x^0\right)^2 - r^2\Delta\theta^2 - \Delta r^2 - \Delta z^2, \ g_{00} = 1 + \frac{2V}{c^2} \qquad (2.6a)$$

Hence, the interval induced on the sub-manifold P_L has the following expression:

$$\Delta\bar{s}^2 = g_{00}\left(\Delta x^0\right)^2 - L^2\Delta\theta^2 \qquad (2.6b)$$

We infer that P moves as a free particle in a two dimensional manifold P_L. Each point of P_L is determined by the time-coordinate: $x^0 = ct$ and the angle-coordinate: θ

According to 2.6b, the matrix of the metric tensor $\left[\bar{g}_{\mu\nu}\right]$ induced on P_L in the coordinate system $\left(ct, \theta\right)$ is:

$$\left[\bar{g}_{\mu\nu}\right] = \begin{pmatrix} g_{00} & 0 \\ 0 & -L^2 \end{pmatrix} \qquad (2.7a)$$

Where: $\bar{g}_{00} = g_{00} = 1 + \frac{2V}{c^2}$, $V = -gL\cos\theta$ and $\bar{g}_{11} = -L^2$

The inverse matrix:

$$\left[\bar{g}^{\mu\nu}\right] = \begin{pmatrix} 1/g_{00} & 0 \\ 0 & -1/L^2 \end{pmatrix} \qquad (2.7b)$$

Now, we derive the connection of P_L which is compatible with its metric (2.6b). To this end, we calculate the Christoffel symbols of the connection by the following equations:

$$\bar{\Gamma}_{\kappa\lambda\mu} = \frac{1}{2}\left(-\partial_\kappa \bar{g}_{\lambda\mu} + \partial_\mu \bar{g}_{\kappa\lambda} + \partial_\lambda \bar{g}_{\mu\kappa}\right),\ \kappa,\lambda,\mu = 0,1$$

Where we have established the correspondence: $0 \leftrightarrow x^0$, $1 \leftrightarrow \theta$

$$\bar{\Gamma}_{000} = 0,\ \bar{\Gamma}_{001} = \bar{\Gamma}_{010} = \frac{1}{2}\frac{dg_{00}}{d\theta} = \frac{gL}{c^2}\sin\theta,\ \bar{\Gamma}_{011} = 0$$

$$\bar{\Gamma}_{100} = -\frac{1}{2}\frac{dg_{00}}{d\theta} = -\frac{gL}{c^2}\sin\theta,\ \bar{\Gamma}_{110} = \bar{\Gamma}_{101} = 0,\ \bar{\Gamma}_{111} = 0$$

$$\bar{\Gamma}^{\nu}_{\lambda\mu} = \bar{g}^{\nu\kappa}\bar{\Gamma}_{\kappa\lambda\mu}$$

$$\bar{\Gamma}^{0}_{00} = 0,\ \bar{\Gamma}^{0}_{10} = \bar{\Gamma}^{0}_{01} = \bar{g}^{00}\bar{\Gamma}_{010} = \frac{1}{2}\bar{g}^{00}\frac{d\bar{g}_{00}}{d\theta} = \frac{gL\sin\theta}{c^2 - 2gL\cos\theta},\ \bar{\Gamma}^{0}_{11} = 0$$

$$\bar{\Gamma}^{1}_{00} = \bar{g}^{11}\bar{\Gamma}_{100} = -\frac{1}{2}\bar{g}^{11}\frac{d\bar{g}_{00}}{d\theta} = \frac{g}{c^2 L}\sin\theta,\ \bar{\Gamma}^{1}_{01} = \bar{\Gamma}^{1}_{10} = 0,\ \bar{\Gamma}^{1}_{11} = 0$$

We, finally write down the differential equations of motion 2.2e for a particle P moving on the sub-manifold P_L. As we have already said, we shall use the world time t to express the analytic equations of motion:

From 2.6b, we have:

$$\Delta\bar{s} = c\Delta t\sqrt{g_{00} - \frac{L^2}{c^2}\dot{\theta}^2} = \frac{c\Delta t}{\bar{\gamma}}$$

Where: $\dot{\theta} \underset{def}{=} \dfrac{d\theta}{dt}$, $\bar{\gamma} \underset{def}{=} \left(g_{00} - \dfrac{L^2}{c^2}\dot{\theta}^2\right)^{-1/2}$ and: $g_{00} = 1 - \dfrac{2gL}{c^2}\cos\theta$

Hence, from 2.2e we obtain the equations:

$$\frac{d\bar{\gamma}}{dt} + 2\bar{\gamma}\bar{\Gamma}^{0}_{01}\frac{d\theta}{dt} = 0 \qquad (2.8a)$$

$$\frac{d}{dt}\left(\bar{\gamma}\frac{d\theta}{dt}\right) + c^2\bar{\gamma}\bar{\Gamma}^{1}_{00} = 0 \qquad (2.8b)$$

From 2.8a we result in the conservation-equation:

$$g_{00}\frac{d\bar{\gamma}}{dt} + \bar{\gamma}\frac{dg_{00}}{d\theta}\frac{d\theta}{dt} = 0,\ \text{or:}$$

$$(2.9a)$$

$$\frac{d}{dt}\left(g_{00}\bar{\gamma}\right) = 0$$

Hence, the following quantity determines the energy of the moving particle, which is conserved along its path:

$$E = mc^2\frac{g_{00}}{\sqrt{g_{00} - \frac{L^2}{c^2}\dot{\theta}^2}} \qquad (2.9b)$$

In the non-relativistic limit, that is for $\dfrac{gL}{c^2}, \dfrac{L^2\dot{\theta}^2}{c^2} \to 0$ expand 3.7b in a Taylor series and keep terms up to the first order. The following result is obtained:

$$E \approx mc^2 - mgL\cos\theta + \frac{1}{2}mL^2\dot{\theta}^2 \qquad (2.9c)$$

Apart from the constant term mc^2 which does not affect the equations of the motion, the previous result is identical to the energy of the particle derived in the Newtonian model (paragraph 1, relation 1.6a).

From 2.8b, we imply the equation:

$$\frac{d}{dt}\left(\bar{\gamma}\frac{d\theta}{dt}\right)+\frac{g}{L}\bar{\gamma}\sin\theta=0 \qquad (2.10)$$

Where: $\bar{\gamma}=\left(1-\frac{2gL}{c^2}\left(\cos\theta+\frac{L}{2g}\dot{\theta}^2\right)\right)^{-1/2}$

Assume that for t=0 the deviation angle θ from the vertical direction is $\theta_0>0$ and the angular velocity $\dot{\theta}$ is zero. Then, from the energy conservation (2.9b) we obtain:

$$\dot{\theta}=\sqrt{\frac{2g}{L}\left(\cos\theta-\cos\theta_0\right)\frac{1-\frac{2gL}{c^2}\cos\theta}{1-\frac{2gL}{c^2}\cos\theta_0}} \qquad (2.11a)$$

From 2.11a we infer that for any value of the initial angle $\theta_0\in(-\pi,\pi)$ the values of θ at any time t, are confined in the interval: $\left[-\theta_0,\theta_0\right]$

In the context of the Newtonian model, the corresponding to 2.11a is the equation:

$$\dot{\theta}=\pm\sqrt{\frac{2g}{L}\left(\cos\theta-\cos\theta_0\right)} \qquad (2.11b)$$

In the non-relativistic limit, i.e. for $\frac{2gL}{c^2}\to 0$, $\bar{\gamma}\to 1$ 2.11a converges to 2.11b, and 2.10 converges to 1.6c.

3. Notations on the composition of the simulation

The values of basic constants in the environment of the simulation

In the simulation system of units, the light velocity is c=1; the mass of the particle is m=1 and the radius of the circular curve is L=0.04.

The initial state of the mechanical system in both models is determined by the conditions:

$\theta(0)=\theta_0$, $\dot{\theta}(0)=0$

The range of the crucial parameter $\frac{2gL}{c^2}=0.08g$ varies from 0 to 1, implying that 0<g<12.5.

The mechanical energy of the system is estimated by the equations:

$$E=mc^2\sqrt{1-\frac{2gL}{c^2}\cos\theta_0}=mc^2\sqrt{1-0.08g\cos\theta_0} \qquad (3.1a)$$

$$mc^2\geq E\geq mc^2\sqrt{1-\cos\theta_0} \qquad (3.1b)$$

How the readings of the clocks in each model are related?

In the absence of the gravitational field, the reference system ($O;t,x,y$) in the virtual environment of the simulation is Cartesian and inertial for the Newtonian model as for the relativistic one. Time t is universal for both models: we have placed two similar chronometers at the space origin O of each model; the time-laps and the readings of both chronometers are identical.

In the presence of a static gravitational field, the time in the relativistic model is world-like, again: we are possible to synchronize the clocks at every space-point of the space-time continuum so that they show the same reading with the clock at the origin O (paragraph 2).

The system-evolution with time

The state of the mechanical system is determined by one degree of freedom: the angle θ specifying the deviation of the particle from the vertical direction, on the circular orbit.

Although the differential equation of motion is of second order, we obtain a first integral of the motion expressed by equations 2.11a and b, respectively.

The radial velocity $v_R=L\dot{\theta}$ of the particle as a function of the angle θ is calculated by the energy conservation and the initial conditions:

$$\frac{g_{00}(\theta)}{\sqrt{g_{00}(\theta)-\frac{v_R^{\;2}}{c^2}}}=\sqrt{g_{00}(\theta_0)} \qquad (3.2)$$

The quantity under the squared root must be greater than or equal to zero for every $\theta \in [-\theta_0, \theta_0]$. Hence v_R satisfies the inequality:

$$\frac{v_R^{\;2}}{c^2} \le 1-\frac{2gL}{c^2} \qquad (3.3)$$

From 3.2, we result that the magnitude of the radial velocity varies with angle θ according to the relationship:

$$\frac{v_R}{c}=\sqrt{g_{00}(\theta)\left(1-\frac{g_{00}(\theta)}{g_{00}(\theta_0)}\right)} \qquad (3.4)$$

Which are the extreme-points of 3.4?
We calculate them by solving the equation:

$$\frac{d}{d\theta}\left(\frac{v_R^{\;2}}{c^2}\right)=0$$

We find that:

$$\frac{d}{d\theta}\left(\frac{v_R^{\;2}}{c^2}\right)=\frac{dg_{00}(\theta)}{d\theta}\left(1-\frac{2g_{00}(\theta)}{g_{00}(\theta_0)}\right)$$

Hence, the velocity-magnitude is an extremum if the following conditions are fulfilled:

$$\frac{dg_{00}(\theta)}{d\theta}=0 \Rightarrow \sin\theta=0 \Rightarrow \theta=0$$

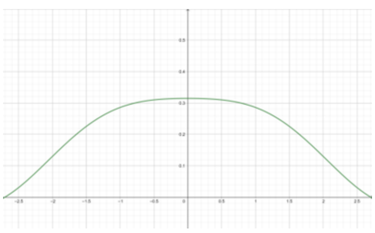
Or:

$$1-\frac{2g_{00}(\theta)}{g_{00}(\theta_0)}=0 \Rightarrow \cos\theta=\frac{c^2}{4gL}+\frac{1}{2}\cos\theta_0$$

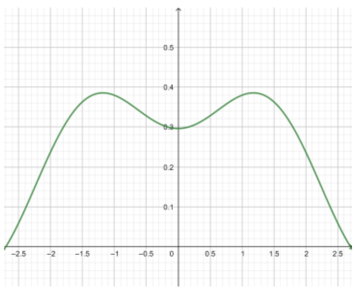
The second case is possible to be fulfilled if the gravitational field is strong enough so that the following condition is satisfied:

$$\frac{2gL}{c^2}>\frac{1}{2-\cos\theta_0} \qquad (3.4)$$

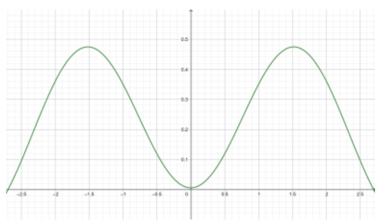
In the following graphs, we can see the variation of the quantity $\frac{v_R^{\;2}}{c^2}$ versus θ for three different values of the parameter $\frac{2gL}{c^2}$



$\frac{2gL}{c^2}=0.3$, $\theta_0=2.7rad$



$\frac{2gL}{c^2}=0.6$, $\theta_0=2.7rad$



$\frac{2gL}{c^2}=0.994$, $\theta_0=2.7rad$

Confirm that if $\frac{2gL}{c^2}\to 1$ the magnitude of the radial velocity converges to zero: the particle is "trapped" by the gravitational field. At first sight, we could say that we have a violation of the energy-conservation. But if we apply carefully the energy-conservation equation (3.2), we can see that this particle-trapping is actually in agreement with this:

Set: $\frac{2gL}{c^2} = 1 - \varepsilon$, $\varepsilon \to 0$

Then, from 3.2 we result in the consequent equations:

$$\frac{g_{00}(\theta)}{\sqrt{g_{00}(\theta) - \frac{v_R^{\,2}}{c^2}}} = \sqrt{g_{00}(\theta_0)}$$

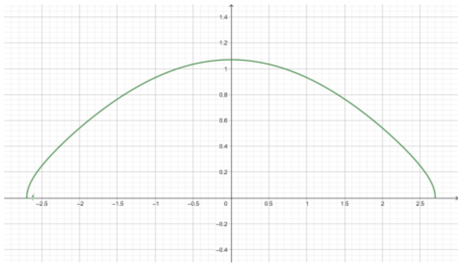
$$\left.\frac{v_R^{\,2}}{c^2}\right|_{\theta=0} = g_{00}(\theta)\left.\left(1 - \frac{g_{00}(\theta)}{g_{00}(\theta_0)}\right)\right|_{\theta=0} = \varepsilon\left(1 - \frac{\varepsilon}{1-(1-\varepsilon)\cos\theta_0}\right) \approx \varepsilon\left(1 - \varepsilon\left(\frac{1}{1-\cos\theta_0} - \frac{\varepsilon\cos\theta_0}{(1-\cos\theta_0)^2}\right)\right) \approx \varepsilon$$

$$v_R(0) \approx c\sqrt{\varepsilon} \to 0 \qquad (3.5)$$

In the context of the Newtonian model, the radial velocity of the particle is given as a function of the deviation angle by the relation:

$$v_R = \sqrt{2gL(\cos\theta - \cos\theta_0)} \qquad (3.6)$$

At $\theta = 0$ function 3.6 has a maximum. Its graph, for the initial angle: $\theta_0 = 2.7rad$ is depicted by the nearby figure.



$\frac{2gL}{c^2} = 0.6$, $\theta_0 = 2.7rad$

4. Activities

1) In the virtual environment of the simulation, calculate the differences in the period of the motion according to each model for the following values of the parameters:
   $a: g = 7,\ \theta_0 = \pi/3rad$ – $b: g = 7,\ \theta_0 = 3.1rad$ – $c: g = 7,\ \theta_0 = 0.2rad$
   $d: g = 1,\ \theta_0 = \pi/3rad$ – $e: g = 1,\ \theta_0 = 0.2rad$ – $f: g = 12.4,\ \theta_0 = 0.2rad$
   $g: g = 12.4,\ \theta_0 = 1.05rad$ – $h: g = 12.4,\ \theta_0 = 3rad$
   Discuss the results in the context of each theoretical model.
2) Formulate the energy-conservation equation for the mechanical system in the relativistic model. In the virtual environment calculate the mechanical energy of the particle in the following cases: a) in the initial condition, b) when the bead passes by the vertical. Apply for the subsequent values of the parameters:
   $a: g = 7,\ \theta_0 = \pi/3rad$
   $b: g = 7,\ \theta_0 = 3.1rad$
   $c: g = 12.44,\ \theta_0 = 0.2rad$
   $d: g = 12.44,\ \theta_0 = 3rad$
   In every case check the fulfillment of the energy-conservation.

3) Repeat activity 2 for the Newtonian model.

4) Set $g = 12,\ \theta_0 = 2.1rad$ and run the simulation. In the virtual environment of the relativistic model, calculate the values of the deviation angle corresponding to the extrema of the particle's velocity magnitude. Calculate the corresponding extrema of the velocity.
   According to the theoretical model, if the condition $\frac{2gL}{c^2} > \frac{1}{2-\cos\theta_0}$ is fulfilled then, the values of the deviation angle corresponding to the extrema of the particle-velocity magnitude are given by the equation:

$$\cos\theta = \frac{c^2}{4gL} + \frac{1}{2}\cos\theta_0$$

Check if the values you obtained in the virtual environment satisfy the previous relationships.

Repeat the same procedure for: $g = 12.44$, $\theta_0 = 3.1 rad$

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