The model of the linear oscillator in Special Relativity and Newtonian Mechanics

Kostas Papamichalis Dr. of Theoretical Physics (Athens University - Greece)

Synopsis

In this work, we define, study and simulate the linear oscillator in two contexts: a) Relativistic Mechanics and b) Newtonian Mechanics. We describe the relativistic and the Newtonian model and derive the corresponding equations of the motion in an inertial reference frame. The time *t* in the Newtonian model is identical with the world time in the relativistic model. The motion differential equations are expressed with free parameter the common time *t*. The initial conditions and the parameters of the oscillator which determine uniquely the motion in each model, have been chosen so that in the non-relativistic limit the predictions of the Newtonian and the relativistic model are identical.

The motion of the Newtonian and the relativistic oscillator are simulated in the virtual environment of the application. The user is free to change the mechanical energy of the system and calculate the variations to the period and the frequency of the motion caused by his choice. In addition, he can watch in real time, the time-position and the position-velocity graph and compare the predictions of each model.

Key words and relations

Inertial system of coordinates in a Minkowski space(1,3,5) - Proper time - World time(3,5) - Minkowski forces(1,3) - The motion equations of a particle in a Minkowski space(1,3) - Relativistic oscillator(1,2,3) - The 2nd Newton-law(2) - Newtonian oscillator(2)

1. The relativistic model of the linear oscillator(3)

In this paragraph we write the motion differential equations of a particle P, in a Minkowski space, and we specialize their form in the case of an inertial, Cartesian reference frame (*O*,x). We formulate the relation of the P's proper time τ with the world time *t* of the frame (*O*,x), and we write down the motion equations, by using *t* as the free parameter. We derive the general constraints that a Minkowski force must satisfy, as well as its form in the case that this force comes from a scalar potential.

We examine the behaviour of a Minkowski force in the non-relativistic limit and we derive the relation of the relativistic scalar potential with the corresponding Newtonian potential. Then, we define the linear oscillator in a certain inertial reference system of coordinates and write down the differential equations that describe its motion. We compare the predictions of the relativistic model of the linear oscillator with the corresponding predictions of the Newtonian model.

A relativistic model that describes the motion of a particle P of mass *m*>0 in a Minkowski space *M*, is determined by the analytic expression of the force four-vector *K*, acting on P. This analytic expression is specified in a certain reference system (*O*,x). Then, the world line *C* of P in (*O*,x) is a solution of the differential equations (1,3,5):

$$m\frac{D_{\Delta C}U}{D\tau}=K \text{ ή: } m\frac{D_{\Delta C}{}^{\mu}U}{D\tau}=K^{\mu}\,,\ \mu=0,1,2,3 \qquad (1.1)$$

…where τ symbolizes the proper time and *U* the four-velocity of P, along its world line *C*(1,2,3). With $D_{\Delta C}U$, we symbolize the covariant differential(3,4) of the P's four-velocity along the curve *C*.

In our model, we consider that (*O*,x) is an inertial, Cartesian system of coordinates(3). The metric tensor of the Minkowski space-time continuum *M* with respect to (*O*,x), is determined by the matrix:

$$\left[\eta^{\mu\nu}\right]=\left[\eta_{\mu\nu}\right]^{-1}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

The infinitesimal interval Δs between two neighboring points of the curve *C*, with coordinates:

$X \leftrightarrow \left(x^0, x^1, x^2, x^3\right)$ and $Y \leftrightarrow \left(x^0+\Delta x^0, x^1+\Delta x^1, x^2+\Delta x^2, x^3+\Delta x^3\right)$

...is calculated by the expressions[1,3,5]:

$$\Delta s = \sqrt{\eta_{\mu\nu}\Delta x^{\mu}\Delta x^{\nu}} = \sqrt{(c\Delta t)^2-(\Delta x^1)^2-(\Delta x^2)^2-(\Delta x^3)^2} = c\Delta t\sqrt{1-\frac{v^2}{c^2}}$$

$$= \frac{1}{\gamma(v)}c\Delta t\,,\ \gamma(v) \underset{def}{=} \left(1-\frac{v^2}{c^2}\right)^{-1/2},\ \ v = \sqrt{\left(\frac{dx^1}{dt}\right)^2+\left(\frac{dx^2}{dt}\right)^2+\left(\frac{dx^3}{dt}\right)^2} \qquad (1.2a)$$

In 1.2a, we have set: $\Delta x^0 = c\Delta t$ where c is the light velocity and t the world time[3,5] in (O,x), which is being measured by synchronized similar chronometers placed at the space-points of (O,x)[3,5].

The proper time of P along its world line C, is related with the world time by the equation[1,5]:

$$\Delta\tau = \frac{1}{\gamma(v)}\Delta t \qquad (1.2b)$$

In the inertial, Cartesian system (O,x), the covariant differential $D_{\Delta C}U$ is identical with the directional differential $d_{\Delta C}U$ [3,4]. Hence: $D_{\Delta C}U=d_{\Delta C}U$.

The components of the four velocity U and its norm are given by the relations:

$$U^{\mu} = \frac{dx^{\mu}}{d\tau} = \gamma(v)\frac{dx^{\mu}}{dt}\,,\ U^0 = \gamma(v)c\,,\ U^j = \gamma(v)v^j\,,\ j=1,2,3 \qquad (1.3a)$$

$$\|U\|^2 = \langle U,U\rangle = \eta_{\mu\nu}U^{\mu}U^{\nu} = \gamma^2\left(c^2-\vec{v}\cdot\vec{v}\right) = c^2 \qquad (1.3b)$$

From 1.3b and the motion equations 1.1, we imply that the Minkowski force must satisfy the conditions:

$$\langle K,U\rangle = m\left\langle\frac{DU}{D\tau},U\right\rangle = m\frac{1}{2}\frac{D}{D\tau}\langle U,U\rangle = 0 \Rightarrow \left\{\langle K,U\rangle = 0\,,\ K^0c-\sum_{j=1}^{3}K^jv^j = 0\right\} \qquad (1.4)$$

We deduce that the motion of the particle P in the coordinate system (O,x) is described by the equations 1.1 and the constraints 1.4. We choose as free variable the world time t of (O,x), and we write:

$$\gamma\frac{d}{dt}\left(m\gamma c\right) = K^0 \qquad (1.5a)$$

$$\gamma\frac{d}{dt}\left(m\gamma v^j\right) = K^j\,,\ j=1,2,3 \qquad (1.5b)$$

$$\eta_{\mu\nu}K^{\mu}U^{\nu} = 0 \qquad (1.5c)$$

In the non-relativistic limit, 1.5b converges to the 2nd law of Newton:

$$\frac{v^2}{c^2}\to 0 \Rightarrow \frac{d}{dt}\left(mv^j\right) = \frac{1}{\gamma}K^j\,,\ j=1,2,3$$

The Newtonian force acting on the particle P is related to the Minkowski force with the expression:

$$F^j = \lim_{\frac{v^2}{c^2}\to 0}\frac{1}{\gamma}K^j\,,\ j=1,2,3$$

Minkowski force coming from a scalar potential

In the references 1 and 3 has been shown that with respect to the inertial Cartesian coordinate system (O,x), any vector field with components :

$$K^{\mu} = \frac{dU^{\mu}}{d\tau}V(\mathrm{x}) + B^{\mu} - \frac{1}{c^2}\langle B,U\rangle U^{\mu} \qquad (1.6)$$

where: $B^{\mu} = \eta^{\mu\kappa}\partial_{\kappa}V(\mathrm{x})$ and $\partial_0 V(\mathrm{x}) = 0$

...satisfies the restraints 1.4 or 1.5; hence it is eligible to get the role of a Minkowski force. One can verify the truth of the following equations:

$$\eta_{\mu\nu}K^{\mu}U^{\nu} = \eta_{\mu\nu}\left(\frac{dU^{\mu}}{d\tau}V(\mathrm{x}) + B^{\mu} - \frac{1}{c^2}\langle B,U\rangle U^{\mu}\right)U^{\nu} = V(\mathrm{x})\eta_{\mu\nu}\frac{dU^{\mu}}{d\tau}U^{\nu} + \eta_{\mu\nu}B^{\mu}U^{\nu} - \frac{1}{c^2}\langle B,U\rangle\eta_{\mu\nu}U^{\mu}U^{\nu} =$$

$$= V(\mathrm{x})\frac{d}{d\tau}\left(\eta_{\mu\nu}U^{\mu}U^{\nu}\right) + \eta_{\mu\nu}B^{\mu}U^{\nu} - \langle B,U\rangle = 0$$

In the case of a four-force given by 1.6, its time and space components are given by the relations:

$$K^0 = -\frac{1}{c}\gamma\frac{d}{dt}(\gamma V(\vec{r}))$$

$$K^j = \eta^{jk}\partial_k V - \frac{1}{c^2}\gamma\frac{d}{dt}\left(\gamma V v^j\right)$$

...and the motion equations 1.5a, 1.5b take the form:

$$\gamma\left(mc^2 + V\right) = E \;\left(=\text{constant}\right) \qquad (1.6a)$$

$$\gamma\frac{d}{dt}\left(\left(m + \frac{1}{c^2}V\right)\gamma v^j\right) = -\partial_j V \qquad (1.6b)$$

1.6a declares that the quantity E, which is defined as "the mechanical energy of the system", is conserved along the world line of P. 1.6b is the generalization of the Newton's 2nd law. In the non-relativistic limit, i.e. for $\frac{\vec{v}^2}{c^2} \to 0$ και $\frac{V(\vec{r})}{mc^2} \to 0$ 1.6b converges to the equation:

$$\frac{d}{dt}\left(mv^j\right) = -\partial_j V = F_{(N)j}$$

...from which, we infer that the analytic form of the function V is the same with the potential energy of P in the correspondent Newtonian model. According to this remark, we define the "relativistic oscillator" as the mechanical system which is determined by the following conditions.

Definition of a linear oscillator in a Minkowski space

A) There exists an inertial, Cartesian coordinate coordinate system (O,x) of the Minkowski space M, such that the motion of a particle P with mass m>0 is determined by the equations 1.6a and 1.6b.

B) The potential energy V is determined in (O,x) by the analytic expression:

$$V(\mathrm{x}) = \frac{1}{2}kx^2 \text{ where: } ct \equiv x^0,\; x \equiv x^1,\; y \equiv x^2,\; z \equiv x^3 \qquad (1.7)$$

$$\mathrm{x} \equiv \left(x^0, x^1, x^2, x^3\right) \equiv (ct, x, y, z)$$

The time coordinate t is the world time(1,3) in the coordinate system (O,x).

By conditions A and B we infer that the motion equations 1.6a and 1.6b take the form:

$$\gamma\left(mc^2 + \frac{1}{2}kx^2\right) = E,\; \gamma\frac{d}{dt}\left(\left(m + \frac{k}{2c^2}x^2\right)\gamma v_x\right) = -kx \qquad (1.8)$$

$$\gamma\frac{d}{dt}\left(\left(m + \frac{k}{2c^2}x^2\right)\gamma v_y\right) = 0,\; \gamma\frac{d}{dt}\left(\left(m + \frac{k}{2c^2}x^2\right)\gamma v_z\right) = 0$$

Assume that the world line of P satisfies the initial conditions:

$$x(0) = x_0,\; v_x(0) = v_0,\; v_y(0) = v_z(0) = 0 \qquad (1.9)$$

From 1.8 and 1.9, we imply the equations:

$$\left(m + \frac{k}{2c^2}x^2\right)\gamma v_y = C_y = \text{const.},\; \left(m + \frac{k}{2c^2}x^2\right)\gamma v_z = C_z = \text{const.}$$

…and:

$$C_y = \left(m + \frac{k}{2c^2} x_0{}^2\right)\gamma_0 v_y(0) = 0,\ C_z = \left(m + \frac{k}{2c^2} x_0{}^2\right)\gamma_0 v_z(0) = 0$$

...from which, we deduce that: $v_y(t) = v_z(t) = 0$ for every t. Hence, the particle P moves on the Ox axis of the coordinate system and its world line is a solution of the differential equations:

$$\gamma\left(mc^2 + \frac{1}{2}kx^2\right) = E \qquad (1.10a)$$

$$\gamma\frac{d}{dt}\left(\left(m + \frac{k}{2c^2}x^2\right)\gamma v_x\right) = -kx \qquad (1.10b)$$

A further analysis of the oscillator's motion equations

1.10a denotes the conservation of the oscillator's mechanical energy along its world line:

$$\gamma\left(mc^2 + \frac{1}{2}kx^2\right) = E = \frac{mc^2 + \frac{1}{2}kx_0{}^2}{\sqrt{1 - \frac{v_0{}^2}{c^2}}},\ \gamma = \gamma(v_x) = \left(1 - \frac{v_x{}^2}{c^2}\right)^{-1/2} \qquad (1.11)$$

On the other hand, from 1.10a and 1.10b the following equation is implied:

$$\gamma\frac{E}{mc^2}\frac{dv_x}{dt} = -\frac{k}{m}x \qquad (1.12)$$

We set:

$$E = mc^2 + w \Rightarrow \frac{w}{mc^2} = \frac{1 + \frac{kx_0{}^2}{2mc^2}}{\sqrt{1 - \frac{v_0{}^2}{c^2}}} - 1 \Rightarrow 1 + \frac{w}{mc^2} = \gamma\left(1 + \frac{kx^2}{2mc^2}\right) \qquad (1.13)$$

...and we assume that at t=0 it holds: $x(0) = x_0,\ v_x(0) = v_0 = 0$

From 1.13 and 1.10a we obtain the relation:

$$w = \frac{kx_0{}^2}{2} = mc^2\left(\gamma(v_x)\left(1 + \frac{kx^2}{2mc^2}\right) - 1\right) \qquad (1.14)$$

...and the motion equation 1.12, takes the form:

$$\left(1 + \frac{w}{mc^2}\right)\gamma\frac{dv_x}{dt} = -\frac{k}{m}x \Rightarrow \frac{dv_x}{dt} = -\frac{1 + \frac{kx^2}{2mc^2}}{\left(1 + \frac{w}{mc^2}\right)^2}\frac{k}{m}x \Rightarrow \frac{dv_x}{dt} = -\frac{1 + \frac{kx^2}{2mc^2}}{\left(1 + \frac{kx_0{}^2}{2mc^2}\right)^2}\frac{k}{m}x \qquad (1.15)$$

Remark 1: Let us solve 1.14 for $v_x{}^2$:

$$\frac{v_x{}^2}{c^2} = 1 - \frac{\left(1 + \frac{kx^2}{2mc^2}\right)^2}{\left(1 + \frac{kx_0{}^2}{2mc^2}\right)^2}$$

The max value of $v_x{}^2$ is accomplished for x=0: $\frac{v_{max}{}^2}{c^2} = 1 - \frac{1}{\left(1 + \frac{kx_0{}^2}{2mc^2}\right)^2}$

...from which, it is implied that the condition $|v_{max}|<c$ is satisfied for every value of the initial mechanical energy $w = \frac{kx_0{}^2}{2}$

Remark 2: Again from 1.14, it is deduced that: $\left(1+\frac{kx^2}{2mc^2}\right)^2 = \left(1-\frac{v_x^{\ 2}}{c^2}\right)\left(1+\frac{kx_0^{\ 2}}{2mc^2}\right)^2$ from which, we can confirm that the max value of x is accomplished for $v_x^2=0$:

$x_{\max} = |x_0| \Rightarrow -|x_0| \le x \le |x_0|$

Remark 3: For $\frac{w}{mc^2} = \frac{kx_0^{\ 2}}{2mc^2} << 1$ equation 1.15 is approximated by the following:

$$\frac{dv_x}{dt} = -\frac{k}{m}x$$

...which is identical with the motion differential equation of a harmonic oscillator in the context of the Newtonian Mechanics, with frequency: $\omega_N = \sqrt{\frac{k}{m}}$

The trajectory of a Newtonian oscillator with initial conditions $x_N(0)=x_0$ and $v_x(0)=0$, is determined by the analytic expression: $x_N = x_0 \cos\omega_N t$

In the followings, we calculate the relativistic correction of the Newtonian path and of its period, in terms up to the first order of the quantity $\frac{w}{mc^2}$

We set: $h \equiv \frac{w}{mc^2} = \frac{kx_0^{\ 2}}{2mc^2} << 1$

We assume that the analytic expression of the path in first order of h, which is a solution of 1.15, with initial conditions $x(0) = x_0,\ v_x(0) = v_0 = 0$ is identified by the function:

$$x = x_0\left(\cos\omega_N t + hf(t)\right) \text{ όπου: } f(0) = 0 \text{ και } \dot{f}(0) = 0 \qquad (1.16a)$$

$$(1.16a) \Rightarrow v_x = \dot{x} = x_0\left(-\omega_N \sin\omega_N t + h\dot{f}(t)\right),\ \dot{v}_x = \ddot{x} = x_0\left(-\omega_N^{\ 2}\cos\omega_N t + h\ddot{f}(t)\right) \qquad (1.16b)$$

From 1.14 and the previous relations 1.16a, 1.16b, we obtain the approximate expression:

$$v_x^{\ 2} \approx \frac{k}{m}\left(x_0^{\ 2} - x^2\right) - \frac{k}{m}h\left(\frac{3}{2}x_0^{\ 2} - 2x^2 + \frac{x^4}{2x_0^{\ 2}}\right) \qquad (1.17)$$

(Notice that the first term at the right part of 1.17 corresponds to the prediction of the Newtonian model)

From 1.17 one confirms that for $|x|=x_0$ the velocity v_x is zero. Hence, the motion is confined in the interval $[-x_0,x_0]$, and it is periodic with period T, which is to be calculated by the equation:

$$T = 2\int_{-x_0}^{x_0}\frac{dx}{v_x} \qquad (1.18)$$

By using 1.17, we approximate the integral at the right part of 1.18, as follows:

$$v_x^{-1} \approx \left(\frac{m}{2w}\right)^{1/2}\left(1-y^2\right)^{-1/2}\left(1+\frac{h}{4}\left(3-y^2\right)\right) \text{ where: } y = \frac{x}{x_0} \qquad (1.18a)$$

$$T \approx 4\sqrt{\frac{m}{k}}\left[\int_0^1 dy\left(1-y^2\right)^{-1/2} - \frac{1}{4}\frac{w}{mc^2}\int_0^1 dy\left(1-y^2\right)^{-1/2}\left(y^2-3\right)\right] \qquad (1.18b)$$

We can easily find the values of the integrals in the right part of 1.18b and conclude to the expressions:

$$T \approx T_N\left(1+\frac{5w}{8mc^2}\right) \text{ where: } T_N = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{mx_0^{\ 2}}{2w}} \qquad (1.18c)$$

$$\omega = \frac{2\pi}{T} \approx \omega_N\left(1+\frac{5w}{8mc^2}\right)^{-1} \approx \omega_N\left(1-\frac{5w}{8mc^2}\right) \text{ where: } \omega_N = \frac{2\pi}{T_N} \qquad (1.18d)$$

Dependence of the relativistic oscillator differential equations of motion from the mechanical energy *w*

The world line of P with initial conditions $x(0)=x_0$, $v_x(0)=0$ and initial mechanical energy $w>0$, is a solution of the differential equation:

$$\frac{dv_x}{dt} = -\frac{1+\frac{w}{mc^2 x_0{}^2}x^2}{\left(1+\frac{w}{mc^2}\right)^2}\frac{2w}{mx_0{}^2}x \qquad (1.19)$$

…or, equivalently:

$$\frac{dv_x}{dt} = -\frac{w}{\frac{mx_0{}^2}{2}\left(1+\frac{w}{mc^2}\right)}\frac{1}{\gamma(v_x)}x \qquad (1.20a)$$

...where (see relation 1.10a):

$$\frac{1}{\gamma(v_x)} = \left(1-\frac{v_x{}^2}{c^2}\right)^{1/2} = \frac{1+\frac{w}{mc^2}\frac{x^2}{x_0{}^2}}{1+\frac{w}{mc^2}} \qquad (1.20b)$$

We have set: $h \equiv \frac{w}{mc^2}$ Hence, from 1.20a and b we deduce the equations:

$$\frac{dv_x}{dt} = -\frac{2w}{mx_0{}^2}\frac{1+h\frac{x^2}{x_0{}^2}}{(1+h)^2}x \qquad (1.20c)$$

$$v_x = c\left(1-\left(\frac{1+h\,x^2/x_0{}^2}{1+h}\right)^2\right)^{1/2} \qquad (1.20d)$$

The period of the oscillation is calculated by the 1.18:

$$T = 2\int_{-x_0}^{x_0}\frac{dx}{v_x} = \frac{2x_0}{c}\int_{-1}^{+1}dy\left(1-\left(\frac{1+h\,y^2}{1+h}\right)^2\right)^{-1/2} \qquad (1.20e)$$

In the non-relativistic limit $\left(h \equiv \frac{w}{mc^2} \to 0\right)$ 1.20c converges to the equation:

$$\frac{dv_x}{dt} = -\frac{2w}{mx_0{}^2}x$$

...and by using 1.14, we confirm that: $\frac{dv_x}{dt} = -\frac{k}{m}x$ which is the well-known motion-equation of a Newtonian oscillator.

Calculation of the period for extremely large values of the oscillator's mechanical energy

Assume that the mechanical energy w of the relativistic oscillator is much larger than the internal energy mc^2 of the oscillating particle P:

$\frac{w}{mc^2} \to +\infty$ In that case, we set:

$$\bar{h} \underset{\text{ορισμ.}}{=} \left(\frac{w}{mc^2}\right)^{-1}$$

Hence: $\frac{w}{mc^2} \to +\infty \Rightarrow \bar{h} \to 0$

We derive the limiting expression of v_x, as a function of x, by using 1.20b, and we calculate the period T, from 1.18:

$$(1.20b) \Rightarrow \left(1-\frac{v_x{}^2}{c^2}\right)^{1/2} = \frac{1+\dfrac{w}{mc^2}\dfrac{x^2}{x_0{}^2}}{1+\dfrac{w}{mc^2}} \Rightarrow 1-\frac{v_x{}^2}{c^2} = \frac{\left(\bar{h}+\dfrac{x^2}{x_0{}^2}\right)^2}{\left(\bar{h}+1\right)^2} \Rightarrow \lim_{\bar{h}\to 0} v_x = c\left(1-\left(\frac{x}{x_0}\right)^4\right)^{1/2}$$

$$(1.18) \Rightarrow T = 2\int_{-x_0}^{x_0}\frac{dx}{v_x} = 4\int_0^{x_{\max}}\frac{dx}{v_x} \to \frac{4x_0}{c}\int_0^1\frac{dq}{\left(1-q^4\right)^{1/2}}$$

The approximate value of the integral at the right part of the previous relation is obtained by using the app "Geogebra", or by a suitable program of the JavaScript. We find that:

$$T \to 5.24\frac{x_0}{c} \qquad (1.21)$$

On the contrary, the period of the motion for extremely large values of the mechanical energy in the Newtonian model, is given by the relations:

$$T_N = 2\pi\sqrt{\frac{m}{k}} = 2\pi x_0\sqrt{\frac{m}{2w}} = \sqrt{2}\pi\frac{x_0}{c}\sqrt{\frac{1}{w/mc^2}} = \sqrt{2}\pi\frac{x_0}{c}\sqrt{\bar{h}}$$

...Hence, in the limit $\bar{h}\to 0$ it is implied that:

$$\lim_{\bar{h}\to 0} T_N = \lim_{\bar{h}\to 0}\left(\sqrt{2}\pi\frac{x_0}{c}\sqrt{\bar{h}}\right) = 0 \qquad (1.22)$$

1. Composition of the virtual environment

From 1.2b we deduce the equation:

$$\frac{v_x{}^2}{c^2} = 1-\left(\frac{1+\dfrac{w}{mc^2}\dfrac{x^2}{x_0{}^2}}{1+\dfrac{w}{mc^2}}\right)^2 \qquad (2.1)$$

...from which it is implied that the extreme values of the velocity are obtained for x=0; v_x takes values in the interval $\left[-V_0, V_0\right]$ where:

$$V_0 = c\sqrt{1-\frac{1}{\left(1+w/mc^2\right)^2}} = c\sqrt{1-\frac{1}{\left(1+h\right)^2}} \qquad (2.2)$$

The equilibrium position of P, is determined by the condition: $dv_x/dt=0$ Hence, from 1.19 it follows that the the position of the equilibrium is at x=0.

Which is the max displacement of P *from its equilibrium position?*
From 1.20b, we imply that the extreme values of x are obtained for v_x=0:

$$1+\frac{w}{mc^2}\frac{x^2}{x_0{}^2} = \left(1-\frac{v_x{}^2}{c^2}\right)^{1/2}\left(1+\frac{w}{mc^2}\right) \Rightarrow \left\{|x| = x_{\max} \Leftrightarrow v_x = 0\right\} \Rightarrow x_{\max} = x_0 \Rightarrow x \in \left[-x_0, x_0\right]$$

In the virtual environment of the simulation, the initial conditions of the oscillator in both models are kept fixed: we assume that at t=0, the particle P is placed at the position $x=x_0$ of the axis Ox of the inertial, Cartesian system (O,x), and its velocity is zero. We simulate the motion of P by using the world time of (O,x) as the free parameter, and we study the variation of the period of the motion as a function of the oscillator's mechanical energy w.
For both, the relativistic and the Newtonian model, the graphs of the position versus time and the velocity versus position are designed in real time. In addition, in the virtual environment the user will find the graph of the period versus the mechanical energy of the oscillator, for both models,

that depicts their different predictions. The graph of the velocity versus position in the relativistic model follows from the energy conservation (relation 2.1):

$$v_x = c\left[1-\left(\frac{1+\frac{w}{mc^2}\frac{x^2}{x_0^2}}{1+\frac{w}{mc^2}}\right)^2\right]^{1/2} \qquad (2.3)$$

In the non-relativistic limit $w/mc^2 \to 0$ 2.3 converges to the corresponding relation predicted by the Newtonian model:

$$v_N = \lim_{\frac{w}{mc^2}\to 0} v_x = \lim_{\frac{w}{mc^2}\to 0} c\left(1-\left(\frac{1+\frac{w}{mc^2}\frac{x^2}{x_0{}^2}}{1+\frac{w}{mc^2}}\right)^2\right)^{1/2} = \lim_{\frac{w}{mc^2}\to 0}\left(c^2-\frac{c^2+2\frac{w}{m}\frac{x^2}{x_0{}^2}+\frac{w^2}{m^2c^2}\left(\frac{x^2}{x_0{}^2}\right)^2}{1+\frac{2w}{mc^2}+\frac{w^2}{m^2c^4}}\right)^{1/2} =$$

$$= \lim_{\frac{w}{mc^2}\to 0}\left(\frac{\frac{2w}{m}+\frac{w^2}{m^2c^2}-2\frac{w}{m}\frac{x^2}{x_0{}^2}-\frac{w^2}{m^2c^2}\left(\frac{x^2}{x_0{}^2}\right)^2}{1+\frac{2w}{mc^2}+\frac{w^2}{m^2c^4}}\right)^{1/2} = \sqrt{\frac{2w}{m}\left(1-\frac{x^2}{x_0{}^2}\right)} = \sqrt{\frac{k}{m}\left(x_0{}^2-x^2\right)}$$

The last equation is identical with the energy conservation in the Newtonian oscillator:

$$\frac{m}{2}v_N{}^2+\frac{k}{2}x^2=\frac{k}{2}x_0{}^2$$

Arithmetic values - units

In the virtual environment we use the system of atomic units. We have chosen: c=1, m=2000 (approximately the mass of the proton) and x_0=10au (atomic units of length).

Calculations - Activities

1. By using the values of the main parameters and the specific features of the virtual environment, prove that:

   a) The period T_N of the Newtonian oscillator is given by the expression: $T_N = 44.4h^{-1/2}$ where: h=w/mc^2

   b) The period T_E of the relativistic oscillator, for h<<1, is given by the expression:

      $T_E \approx 44.4\,h^{-1/2}\left(1+\frac{5}{8}h\right)$

   c) For $h \to +\infty$ the period of the relativistic oscillator tends to the value:

      $T_E \to 5.24\,\frac{x_0}{c}$

2. Find the limiting behaviour of the period T_N of the Newtonian oscillator for $h \to +\infty$

3. Check your previous calculations by running the simulation.
Hint: For the activity 1b set successively: h=0.05, h=0.1, h=0.2. For the activity 1c set successively: h=3, h=5, h=10 (the value h=10 is the max value of h accepted by the application and the value h=0.01 the min value)

4. Notice that the shape of the cyclic particle oscillating according to the relativistic model (blue particle), is changing. Run the simulation and describe how this change takes place. Explain this

phenomenon in the context of the Special Theory of Relativity, and calculate the variation of the particle's diameter which is parallel to the x-axis, as a function of the particle's velocity.

Virtual experiments

Experiment 1
Set *h*=0.1 and run the simulation.
A) By using the window of the virtual motion, the chronometer and the button of the step-evolution, calculate and write down the value of the period of each oscillator.
B) Calculate and write down the period of each oscillator by using the graphs: a) time-position, b) position-velocity, c) energy (*h*)-period (*T*).
Run the simulation as many times as necessary.
C) Do your calculations agree with each-other? If you notice any inconsistency, repeat the virtual experiment and try to find out where is the error.

Experiment 2
Repeat the activities A, B, C, mentioned in experiment 1, by setting *h*=1.

Experiment 3
Repeat the activities A, B, C, mentioned in experiment 1, by setting *h*=10.

References

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4) Physical Mathematics: Kevin Cahill, Cambridge University Press 2013.
5) The Classical Theory of Fields: L.D. Landau, E.M. Lifshitz. Pergamon Press third revised English edition.