Scattering in a central field of forces

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Description and objectives of the application

Key-concepts and relations

Central force field – Scattering – Impact parameter – Scattering angle – Field of a point charge – A model of a neutral atom – A model of a negative ion – Inertial reference frame – Trajectory of a particle – The 2d Newton law – Energy conservation – Angular momentum conservation

Synopsis

In the present simulation is studied the scattering of N positive charged identical -but discrete- particles in a central electrostatic field. Each particle is specified by an index j=0,1…N-1. They all have the same mass m and the same electric charge q. The center of the field is placed at the origin O of the laboratory inertial reference frame Oxyz. The particles are moving on the plane Oxy; their initial positions are determined by the position vectors $\vec{r}_j(0) = \left(-a, b_j, 0\right)$, where a>0 and j=0,1…N-1. The parameter b_j is called the "impact parameter" of the corresponding j-particle. The particles have the same initial velocity: $\vec{v}_j(0) = v_0\hat{x} = \left(v_0, 0, 0\right)$, $v_0 > 0$ j=0,1…N-1.

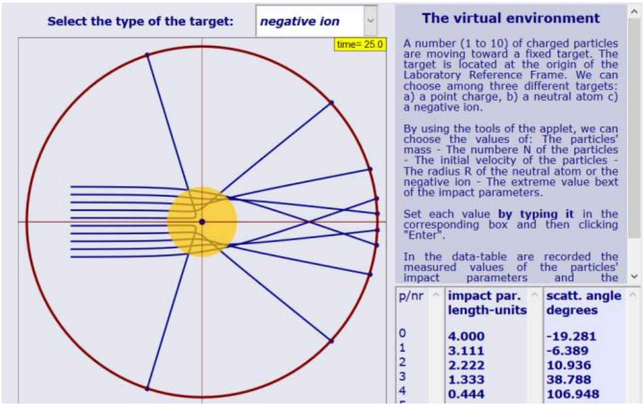

Figure 1: Scattering by a negative ion. The scattering angles and the corresponding impact parameters are recorded by the program, in the data table.

In the virtual environment of the simulation, there is a cyclic screen of center O and radius L. The particles hit on the screen and they leave a small mark on it. The parameter a, as well the radius L of the screen are large enough so that the force acting on the particles at the initial and the final part of the system evolution is negligible. Then, the particles' velocity in the initial and the final part of their motion, is in a good approximation, constant.

The impact parameter b_j of the moving j-particle is defined as the least distance between this particle and the center O of the field if the field is turned off. At t=0, each particle is far enough from O, where the field is weak and the y-coordinate of its initial position is approximately identical to b_j (figure 1).

The angle θ_j formed by the velocity $\vec{v}_j$ and the positive x-axis, when the j-particle has been scattered by the field and it is far from O where the field is negligible, is defined as the "scattering angle" of the j-particle. In the virtual environment of the simulation, the radius L is large enough, so that the force acting on any particle approaching the screen can be considered negligible. The scattering angle θ_j is a function of the impact parameter b_j. **The analytic expression of this function depends on the form of the field**. That means, that if the form of the target producing the field has been changed, the user will observe different values for the scattering angles for the same value of the impact parameter. From the recorded data he or she can make models concerning the structure of the target, and then design and implement the appropriate experiments aiming at the confirmation of his or her conjectures. This logic summarizes the main objective of the application.

Description of the model

Invariant quantities along each particle's trajectory

The charged particles are moving in a conservative, central electrostatic field [1,2]. The energy E_j and the angular momentum $\vec{L}_j$ are invariant along the path of the j-particle; their values are calculated at the initial state of the particle. For every j=0,1, 2...N-1, the following relations are true:

$E_j = \frac{1}{2} m_j {v_0}^2 = \text{constant}$

$\vec{L}_j = m_j \left(-a\hat{x} + b_j\hat{y}\right) \times \left(v_0\hat{x}\right) = m_j v_0 \hat{y} \times \hat{x} = -m_j v_0 \hat{z} = \text{constant}$

where: $\hat{z} = \hat{x} \times \hat{y}$

The angular momentum conservation implies that the motion of each particle is taking place on the Oxy plane which is defined by the particle's initial velocity and the center O of the field. This plane is perpendicular to the angular momentum $\vec{L}_j$ and therefore to the unit vector $\hat{z}$ [1,2].

Selection and adjustment of the parameters in the simulation

The system of units is determined in the program of the simulation. The value of any scalar quantity is determined by a real number and the corresponding simulation unit: "simUnit".

In the virtual environment of the simulation is depicted the plane Oxy where the motion of the particles is taking place. The user can select the common value of the particles' mass in the interval $\left[0.5, 2\right]$ and the number of particles, from 1 to 10. The common value of the particles' electric charge is $q = +2\,\text{simUnit}$

The common value of the initial velocities is chosen in the interval $\left[2, 4\right]$, and the radius R of the atom or ion target, in the interval $\left[2, 5\right]$; if the target is a point charge, the value of R is 'undefined'.

The extreme values b_{ext} of the impact parameters b_j are also selected by the user:

$b_{ext} \in \left[-10, 10\right]$

The program of the simulation configures the initial positions of the charged particles so that their y-coordinates are symmetric with respect to the x-axis and are determined by the equation:

$b_j = b_{ext}\left(1 - \frac{2j}{N-1}\right)$, $j = 0, 1 \ldots N-1$

$b_0 = -b_{ext}$, $b_{N-1} = b_{max}$

The x-coordinate is the same for all particles and equals to -a.

At t=0, the distance between two neighboring particles equals to $\frac{2\,b_{ext}}{N-1}$

The type of target and the field produced by it

The field produced by the fixed target placed at the origin O of the inertial reference frame Oxyz, acts on every charged particle with forces of the general analytic form:

$\vec{F}(\vec{r}) = F(r)\frac{\vec{r}}{r}$

$\vec{r} = \left(x, y\right)$ is the position vector of the particle, and $r = \left(x^2 + y^2\right)^{1/2}$

The form of the function $F(r)$ depends on the target's type. In the present simulation the user can select among three alternatives:

Choice A: point charge

The target is a charged point particle fixed at O. Its electric charge is $Q = +2$ simUnit and the analytic expression of $F(r)$ is given by the relationship:

$$F(r) = \frac{Qq}{r^2} \qquad (1)$$

The value of the electric constant is: $K_{el} = 1\,\text{simUnit}$

Choice B: neutral atom

The target is a neutral atom, determined by the following simplified model:

Around a fixed point core, placed at O and charged with an electric charge $Q = +2\,\text{simUnit}$, there exists a spherical, negatively charged cloud of radius R. Everywhere in the cloud the charge density is constant, but outside the cloud is zero. The total electric charge of the cloud equals to $Q' = -Q$

It is assumed that the charge density of the cloud is not disturbed when the charged particles pass through or near it.

For this model, it is proved (exercise 2) that $F(r)$ is determined by the analytic expression:

$$F(r) = \begin{cases} Qq\left(\frac{1}{r^2} - \frac{r}{R^3}\right) \text{για } 0<r \le R \\ 0 \text{ για } r > R \end{cases} \tag{2}$$

The value of the radius R is controlled by the user.

Choice C: negative ion

The target is a negative ion, determined by the following simplified model:

Around a fixed point core, placed at O and charged with an electric charge $Q = +2\,\text{simUnit}$, there exists a spherical, negatively charged cloud of radius R. Everywhere in the cloud the charge density is constant, but outside the cloud, it is zero. The total electric charge of the negative cloud equals to $Q'' = -2Q$

It is assumed that the charge density of the cloud is not disturbed when the charged particles pass through or near it.

For this model, it is proved (exercise 2) that $F(r)$ is determined by the analytic expression:

$$F(r) = \begin{cases} Qq\left(\frac{1}{r^2} - \frac{2r}{R^3}\right) \text{για } 0<r \le R \\ -\frac{Qq}{r^2} \text{ για } r > R \end{cases} \tag{3}$$

The particles' equations of motion

By applying Newton's second law for every charged particle, we obtain the differential equations of its motion.

$$\begin{aligned} \ddot{x}_j &= \frac{1}{m}F_x\left(x_j, y_j\right) = \frac{1}{m}F(r_j)\frac{x_j}{r_j} \\ \ddot{y}_j &= \frac{1}{m}F_y\left(x_j, y_j\right) = \frac{1}{m}F(r_j)\frac{y_j}{r_j} \end{aligned} \tag{4}$$

The initial conditions that determine uniquely the path of each particle (j=0,1…N-1) are:

$x_j(0) = -a$, $y_j(0) = b_j$, $\dot{x}_j(0) = v_{jx}(0) = v_0$, $\dot{y}_j(0) = v_{jy}(0) = 0$

Annotations on the implementation of the virtual experiments

The data table

The results of the virtual experiments are recorded in a data-table attached to the environment of the simulation. The impact parameter and the corresponding scattering-angle are calculated by the program and recorded in the cells of this table for each particle, in real-time. The scattering angle of the j-particle is the angle θ_j formed by the velocity v_j and the Ox axis at the moment that the j-particle hits the cyclic screen.

The arithmetic solution of the motion-equations

The arithmetic solution of the differential equations of motion for each particle is achieved in the program of the simulation, by using the Euler method.

The particles' positions and velocities are calculated in a sequence of time moments $0, \Delta t, 2\Delta t...$

Symbolize $a_x\left(t_n\right)$, $a_y\left(t_n\right)$ the acceleration coordinates of a certain particle at the time $t_n = n\Delta t$ $n = 0,1,2...$ Then, according to the mean value theorem of the Integral Calculus, the following approximations are valid:

$$v_{xn+1} = v_{xn} + \int_{t_n}^{t_{n+1}} a_x(t)dt \approx v_{xn} + a_x(t_n + 0.5\Delta t)\Delta t \approx v_{xn} + \bar{a}_{xn}\Delta t\,,\ \bar{a}_{xn} \underset{def}{=} a_x(t_n + 0.5\Delta t)$$

$$v_{yn+1} = v_{yn} + \int_{t_n}^{t_{n+1}} a_y(t)dt \approx v_{yn} + a_y(t_n + 0.5\Delta t)\Delta t \approx v_{yn} + \bar{a}_{yn}\Delta t\,,\ \bar{a}_{yn} \underset{def}{=} a_y(t_n + 0.5\Delta t)$$

$$x_{n+1} \approx x_n + v_x(t_n + 0.5\Delta t)\Delta t \approx x_n + \bar{v}_{xn}\Delta t\,,\ \bar{v}_{xn} \underset{def}{=} v_x(t_n + 0.5\Delta t)$$

$$y_{n+1} \approx y_n + v_y(t_n + 0.5\Delta t)\Delta t \approx y_n + \bar{v}_{yn}\Delta t\,,\ \bar{v}_{yn} \underset{def}{=} v_y(t_n + 0.5\Delta t)$$

…where:

$$x_n = x(n\Delta t),\ y_n = y(n\Delta t),\ n = 0,1,2...$$

$$\bar{a}_{xn} = \frac{F_x(\bar{x}_n, \bar{y}_n)}{m} \quad \bar{x}_n = x\left(\left(n + \frac{1}{2}\right)\Delta t\right) \simeq x_n + \frac{1}{2}v_{xn}\Delta t\,,\ \bar{y}_n = y\left(\left(n + \frac{1}{2}\right)\Delta t\right) \simeq y_n + \frac{1}{2}v_{yn}\Delta t$$

$$\bar{v}_{xn} = v_x\left(\left(n + \frac{1}{2}\right)\Delta t\right) \simeq v_{xn} + \frac{1}{2}a_{xn}\Delta t\,,\ a_{xn} = \frac{F_x(x_n, y_n)}{m}$$

$$\bar{v}_{yn} = v_y\left(\left(n + \frac{1}{2}\right)\Delta t\right) \simeq v_{yn} + \frac{1}{2}a_{yn}\Delta t\,,\ a_{yn} = \frac{F_y(x_n, y_n)}{m}$$

Exercises- Activities

Annotations

a) The center of the field is at the origin O of the laboratory inertial frame of reference Oxyz. The radius of the cyclic screen equals to $L = 20\,\text{simUnit}$

b) The value of the electric constant equals to 1: $K_{el} = 1\,\text{simUnit}$

c) The electric charge of each particle is $q = +2\,\text{simUnit}$. The charge of the point-like target is $Q = +2\,\text{simUnits}$; the charge of the negative cloud for the neutral atom is $Q' = Q$, and for the negative ion $Q'' = -2Q$

d) The initial position of the j-particle is $\vec{r}_j(0) = (-a, b_j)$, where: a=15simUnit, and b_j the corresponding impact parameter: $b_j = b_{ext}\left(1 - \frac{2j}{N-1}\right)$, $j = 0,1...N-1$

e) The initial velocity of each particle is $\vec{v}_j(0) = (v_0, 0)$

Exercise 1

A central field of forces with its center at the origin O of the laboratory inertial reference frame is generally determined by the analytic expression:

$$\vec{F}(\vec{r}) = F(r)\frac{\vec{r}}{r} \qquad \text{(E1.1)}$$

$F(r)$ is a function of the position vector length *r*.

a. Show that a central field is conservative. Derive the form of the potential energy of a particle moving in the field E1.1.
b. Show that if a charged point-particle is moving in a central field, its energy and angular momentum are constants of the motion. Show that the path of the particle lies on a constant plane of the lab reference frame.

Exercise 2

A point charge is placed in a spherical homogeneous charge distribution of radius *R* and total charge *Q'*. The distance of the point charge from the center of the distribution is *r*<*R*. Show that:

a. The electrostatic force acted on the point charge from the spherical shell of center O and width $d = R - r$ equals to zero.
b. For the case of the neutral atom model, prove relation 2.
c. For the case of the negative ion model, prove relation 3.

Activities in the virtual environment

Experiment 1: From two virtual scattering experiments of N=10 particles each of mass $m = 1.5\text{simUnit}$ and initial velocity of magnitude $v_0 = 2\text{simUnit}$, resulted in the data recorded in the above tables.

a. Show that the target is a neutral atom.
b. Show that according to the recorded data, the value of atom radius R is in the interval $(3.73, 4.67)$
c. Confirm the recorded data in the virtual environment for $R = 4.5\text{simUnit}$

p/nr	impact par. length-units	scatt. angle degrees
0	4.800	0.000
1	3.733	3.296
2	2.667	12.843
3	1.600	31.456
4	0.533	89.937
5	-0.533	-89.937
6	-1.600	-31.456
7	-2.667	-12.843
8	-3.733	-3.296
9	-4.800	0.000

p/nr	impact par. length-units	scatt. angle degrees
0	6.000	0.000
1	4.667	0.000
2	3.333	6.227
3	2.000	22.693
4	0.667	76.614
5	-0.667	-76.614
6	-2.000	-22.693
7	-3.333	-6.227
8	-4.667	0.000
9	-6.000	0.000

Experiment 2: From a virtual scattering experiment of N=6 particles each of mass $m = 1\text{simUnit}$ and initial velocity of magnitude $v_0 = 2\text{simUnit}$, resulted in the data recorded in the adjacent table.

a. Which is the target structure?
b. Confirm your conclusion for $R = 4\text{simUnit}$

p/nr	impact par. length-units	scatt. angle degrees
0	4.000	-19.281
1	2.400	6.960
2	0.800	70.255
3	-0.800	-70.255
4	-2.400	-6.960
5	-4.000	19.281

Experiment 3: Run two virtual scattering experiments choosing a point charge as a target.
For the first experiment select: $N = 1$, $m = 0.5\text{simUnit}$, $b = 2\text{simUnit}$
For the second experiment select: $N = 1$, $m = 2\text{simUnit}$, $b = 2\text{simUnit}$
Explain theoretically the difference of the scattering angles recorded in the two experiments.

Bibliography

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