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World Lines Transform: the relativistic and the Newtonian point of view

Synopsis
A particle of mass m is moving in a Minkowski space M with constant spatial velocity, with respect to the inertial Cartesian reference system: (O,x), $\mathrm{x}=(x^0,x^1,x^2,x^3)$. We derive the analytic expression of the particle's world line in another inertial Cartesian reference system: (O',x'). We do the same for the case that our particle describes a circular uniform motion in the (O,x) frame. We simulate the motion of the particle for every one of these cases. Then, we repeat this project by studying the corresponding transformations of the particle's paths according to the Newtonian point of view. The user of the simulation compares the data ensuing in the virtual environment of the relativistic and the Newtonian model and writes down his/her conclusions.

Key-concepts
Minkowski space – World line – Four-velocity – Four-momentum – Four-acceleration – Four-force (Minkowski force) – Lorentz transformation –Euclidean space – Galileo transformation

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Paragraph 1: Uniform linear motion in a Minkowski space with respect to two different inertial Cartesian reference systems

Consider a four-dimensional Minkowski space M and an inertial Cartesian reference frame (O,x)
The points of M are determined by the four-tuples of R^4 through a differentiable function [1,5] $\Phi : R^4 \rightleftarrows \Phi(R^4) \equiv M$ The values of Φ are points of an abstract, real vector space of dimension greater or equal to 4.
At every point $P \in M$, the tangent vectors $e_\mu = \partial_\mu \Phi$, $\mu = 0,1,2,3$ of M are defined. They form a set of basis vectors for the tangent space $T_P M$ of M at P and given that (O,x) is inertial and Cartesian, this set is independent of the choice of $P \in M$. The basis vectors e_μ determine the Minkowski inner product in each tangent space $T_P M$, according to the relations:

$$\langle e_\mu, e_\nu \rangle = \eta_{\mu\nu}$$

…where:

$$[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The interval Δs between two infinitesimally close points P and Q of M with coordinates [1,3]:

$$\mathrm{x}_P = (x^0,x^1,x^2,x^3),\ \mathrm{x}_Q = (x^0+\Delta x^0, x^1+\Delta x^1, x^2+\Delta x^2, x^3+\Delta x^3)$$

…is calculated by the equations:

$$\Delta s = \sqrt{\langle e_\mu \Delta x^\mu, e_\nu \Delta x^\nu \rangle} = \sqrt{(\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2}$$

…or, setting: $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$

$$\Delta s = \sqrt{c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2}$$

…where t is the world time measured by all the synchronized clocks setting at the spatial points of the inertial system (O,x) [2,5].

Let P be a particle of mass m moving in the Oxy plane of the inertial system (O,x) with constant (independent of the world time t) spatial velocity $\vec{v}=(v_x,v_y,0)$

The world line of P is the image of the coordinate-space-curve:

$$\mathrm{x}_P(t)=ct\left(1,\frac{v_x}{c},\frac{v_y}{c},0\right) \tag{1.1}$$

…to the Minkowski space M, which is expressed by the composite function: $X_P(t)=\Phi(\mathrm{x}_P(t))$

The infinitesimal interval along the curve X_P is calculated by the equation:

$$\Delta s=\sqrt{\langle \Delta X_P,\Delta X_P\rangle}=c\Delta t\left(1-\frac{v^2}{c^2}\right)^{1/2}$$

The proper time of P is: $\Delta\tau=\frac{1}{c}\Delta s=\frac{1}{\gamma(v)}\Delta t$

…where: $\gamma(v)=\left(1-\frac{v^2}{c^2}\right)^{-1/2},v=\sqrt{{v_x}^2+{v_y}^2}$

The four-velocity and the four-momentum of P are given by the equations:

$$U_P=\frac{dX_P}{d\tau}=e_\mu\frac{dx_P{}^\mu}{d\tau}=\gamma(v)c\left(e_0+e_1\frac{v_x}{c}+e_2\frac{v_y}{c}\right) \tag{1.2}$$

$$P_P=mU_P=e_0m\gamma(v)c+e_1m\gamma(v)v_x+e_2m\gamma(v)v_y\equiv e_0\frac{E}{c}+e_1P_x+e_2P_y \tag{1.3}$$

…where E is the energy of P.

How is the P's world line analytic expression transformed under a homogeneous Lorentz transformation?

Consider a new inertial Cartesian coordinate system (O',x') with its axes parallel to the axes of (O,x) The zeroth events of the two systems are identical i.e., if $x'^\mu=0,\mu=0,1,2,3$ then, it is true that $x^\mu=0,\mu=0,1,2,3$

The spatial origin O of (O,x)-and any point with constant spatial coordinates: $\Delta x^j=0$, $j=1,2,3$ - is moving along the x-axis of (O',x') with velocity $v_0:0<v_0<c$

The analytic expression of this coordinate transformation is determined by the equations [5]:

$$ct'=ct\,L_0^{\ 0}+x\,L_1^{\ 0},\ x'=ct\,L_0^{\ 1}+x\,L_1^{\ 1},\ y'=y,\ z'=z \tag{1.4a}$$

$$L_0^{\ 0}=L_1^{\ 1}=\cosh\theta,\ L_0^{\ 1}=L_1^{\ 0}=\sinh\theta,\ \tanh\theta=\frac{v_0}{c} \tag{1.4b}$$

The world line of O in the (O',x') is determined by the parametric equations:

$$ct'=ct\,L_0^{\ 0},\ x'=ct\,L_0^{\ 1}=v_0t',\ y'=0,\ z'=0 \tag{1.4c}$$

…where t' is defined as the world time in the coordinate system (O',x')

The world line of a fixed space point A in the (O,x) with spatial coordinates $(x_0,y_0,0)$, in the (O',x') is determined by the equations:

$$ct'_A=ct'+x_0\,L_1^{\ 0},\ x'_A=v_0t'+x_0\,L_1^{\ 1},\ y'_A=y_A,\ z'_A=0 \tag{1.4d}$$

The world line of P in the (O',x') frame is accomplished by transforming 1.1 via 1.4a and b:

$$ct'_P=ct\left(L_0^{\ 0}+\frac{v_x}{c}L_1^{\ 0}\right),\ x'_P=ct\left(L_0^{\ 1}+\frac{v_x}{c}L_1^{\ 1}\right),\ y'_P=v_yt,\ z'_P=0 \tag{1.5a}$$

…or choosing the world time t' as the free parameter:

$$ct'_P = ct'\left(1+\frac{v_x}{c}\tanh\theta\right) = ct'\left(1+\frac{v_x v_0}{c^2}\right)$$

$$x'_P = ct'\left(\frac{v_x}{c}+\tanh\theta\right) = t'\left(v_x+v_0\right),\ y'_P = \frac{v_y}{\gamma\left(v_0\right)}t',\ z'_P = 0 \tag{1.5b}$$

The P's spatial velocity coordinates in the $\left(O',\mathrm{x}'\right)$ system, are:

$$v'_x = \frac{\Delta x'_P}{\Delta t'_P} = \frac{v_x + c\tanh\theta}{c+v_x\tanh\theta} = \frac{v_x+v_0}{1+\frac{v_x v_0}{c^2}}$$

$$v'_y = \frac{\Delta y'_P}{\Delta t'_P} = \frac{v_y}{\gamma(v_0)\left(1+\frac{v_x v_0}{c^2}\right)},\ v'_z = 0 \tag{1.5c}$$

The angle formed by the straight path of P and the x-axis in the $\left(O',\mathrm{x}'\right)$ frame is calculated by the equation:

$$\Theta' = \tan^{-1}\left(\frac{v'_y}{v'_x}\right) = \tan^{-1}\left(\frac{1}{\gamma(v_0)}\frac{v_y}{v_x+v_0}\right) \tag{1.5d}$$

The corresponding angle in the $\left(O,\mathrm{x}\right)$ frame is: $\Theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

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Paragraph 2: Uniform circular motion in a Minkowski space with respect to two different inertial Cartesian reference systems

In the present paragraph, we consider that particle P describes a circular uniform motion with respect to the reference frame $\left(O,\mathrm{x}\right)$ The center of the circular path is the spatial point $\left(x_0,y_0,0\right)$ of the reference system $\left(O,\mathrm{x}\right)$ and its radius equals R. The angular velocity ω of the motion is a constant -independent of time.

The parametric equations and the world line of P are:

$$x(t) = x_0 + R\cos\omega t,\ y(t) = y_0 + R\sin\omega t,\ z = 0$$

$$\mathrm{x}_P(t) = \left(ct, x(t), y(t), 0\right),\ X_P(t) = \Phi\left(\mathrm{x}_P(t)\right)\in M \tag{2.1a}$$

$$v_x = \frac{\Delta x}{\Delta t} = -R\omega\sin\omega t,\ v_y = \frac{\Delta y}{\Delta t} = R\omega\cos\omega t,\ v_z = \frac{\Delta z}{\Delta t} = 0$$

The infinitesimal interval and the proper time along the world line are calculated by the equations:

$$\Delta s^2 = c^2\Delta\tau^2 = c^2\Delta t^2\left(1-\frac{{v_x}^2}{c^2}-\frac{{v_y}^2}{c^2}\right) = c^2\Delta t^2\left(1-\frac{R^2\omega^2}{c^2}\right)$$

$$\Delta\tau = \frac{1}{\gamma}\Delta t,\ \gamma = \left(1-\frac{R^2\omega^2}{c^2}\right)^{-1/2}$$

Consider the inertial Cartesian coordinate system $\left(O',\mathrm{x}'\right)$ related to $\left(O,\mathrm{x}\right)$ with the Lorentz transformation 1.4. The world line of P in this coordinate system is expressed as follows:

$$ct'_P = ct\,{L_0}^0 + {L_1}^0\,x(t),\ x'_P = ct\,{L_0}^1 + {L_1}^1\,x(t)$$

$$y'_P = y(t),\ z'_P = 0 \tag{2.1b}$$

…where the analytic expressions of $x(t)$, $y(t)$ are given by 2.1a.

The P's spatial velocity coordinates in the $\left(O',\mathrm{x}'\right)$ system, are:

$$v'_x = \frac{\Delta x'_P}{\Delta t'_P} = \frac{\Delta\left(ct\,{L_0}^1 + {L_1}^1\,x(t)\right)}{\Delta\left(t{L_0}^0 + \frac{1}{c}{L_1}^0\,x(t)\right)} = \frac{v_0 - R\omega\sin(\omega t)}{1-\frac{R\omega v_0}{c^2}\sin(\omega t)} \tag{2.1c}$$

$$v'_y = \frac{\Delta y'_P}{\Delta t'_P} = \frac{\Delta\left(y(t)\right)}{\Delta\left(tL_0{}^0 + \frac{1}{c}L_1{}^0\, x(t)\right)} = \frac{R\omega\cos(\omega t)}{L_0{}^0\left(1 - \frac{R\omega v_0}{c^2}\sin(\omega t)\right)},\ v'_z = 0 \qquad (2.1d)$$

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Paragraph 3: The Newtonian point of view

From the Newtonian point of view, the inertial frames in Cartesian coordinates, are related by the Galileo transformations [1,2]. Hence, relations 1.5 are written (the index 'P' has been dropped):

$$\begin{aligned} &t' = t,\ x' = \left(v_x + v_0\right)t,\ y' = v_y t,\ z' = 0 \\ &v'_x = v_x + v_0,\ v'_y = v_y,\ v'_z = 0 \end{aligned} \qquad (3.1)$$

Relations 2.1, for the circular uniform motion take the form:

$$\begin{aligned} &t' = t,\ x' = v_0 t + x(t),\ y' = y(t),\ z' = 0 \\ &v'_x = v_0 - R\omega\sin\omega t,\ v'_y = R\omega\cos\omega t,\ v'_z = 0 \end{aligned} \qquad (3.2)$$

The user is prompted to study the simulations for each case in the relativistic and the Newtonian context, write down the arising data, and form his/her conclusions.

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