Some Results on Sign Symmetric Matrices

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The structure of talk is the following

The Sign Symmetric Matrices

Consider an $n \times n$ square real matrix A and two subsets α and β of $\{1, 2, \ldots, n\}$ with the same cardinality ($|\alpha| = |\beta|$). We denote by $A[\alpha|\beta]$ the minor with the rows indexed by α and columns indexed by β. If $\alpha = \beta$ the minor is a *principal* minor of A. The matrix A is called *sign symmetric* (*ant sign symmetric*) if

$$A[\alpha|\beta]A[\beta|\alpha] \geq 0 \quad (A[\alpha|\beta]A[\beta|\alpha] \leq 0, \; \alpha \neq \beta),$$

for all α and $\beta \subset \{1, 2, \ldots, n\}$ with $|\alpha| = |\beta|$.

The Stable and $P-$Matrices

A square real matrix A is called a $P-$matrix (P_0-matrix) if all the principal minors of A are positive (nonnegative).
The positive definite matrices and the $M-$matrices belong to the class of $P-$matrices.
A square real matrix A is called *positive stable* or simply *stable* if its eigenvalues have positive real parts or equivalently if its eigenvalues lie in the open right half complex–plane.

Previous works

Many researchers (e.g. Taussky, Carlson) have studied the connection among the class of $P-$matrices with the stability and sign symmetry.
Recently Hershkowitz and Keller (2005) have studied the sign symmetry of basic and shifted basic circulant permutation matrices and have given a simple criterion for [anti] sign symmetric matrices of this class, although they have dealt with 3×3 sign symmetric matrices.

Definitions

An $n \times n$ matrix is called a *basic* $p-$*circulant* permutation matrix if it is defined as follows

$$(C_n^{(p)})_{ij} = \begin{cases} 1 & j = i + p, \text{ if } 1 \leq i \leq n-p \\ 1 & j = i - n + p, \text{ if } n-p < i \leq n \\ 0, & \text{otherwise} \end{cases}$$

The *basic* $p-$*circulant* permutation matrix has the form

$$C_n^{(p)} = \underbrace{\begin{pmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}}_{p}$$

Lemmas, Propositions, Theorems

For the above class of Matrices the following Theorem is valid

Theorem:

Let p be a positive integer, $C_{2n}^{(p)}$ the *basic* $p-$*circulant* permutation matrix, with $g.c.d.(p,n)=1$, and α, β different nonempty subsets of $\{1,2,\ldots,2n\}$ of the same cardinality. The product $C_{2n}^{(p)}[\alpha|\beta]C_{2n}^{(p)}[\beta|\alpha]\neq 0$ if and only if

$$\{\alpha,\beta\}=\{\{1,3,\ldots,2n-1\},\{2,4,\ldots,2n\}\}. \tag{1}$$

Lemmas, Propositions, Theorems

Let p be even. In this case, since $\alpha \neq \beta$ and number 1 is located in positions with only odd or even indices, we have $C_{2n}^{(p)}[\alpha|\beta] = 0$. Also, if $\alpha = \beta = \{1, 2, \ldots, 2n\}$ then $C_{2n}^{(p)}[\alpha|\beta] = 1$ and so the matrix $C_{2n}^{(p)}$ is sign symmetric.
Let p be odd.. In this case the minors have the form:

$$C_{2n}^{(p)}[\alpha|\beta] = \left| \begin{matrix} \underbrace{\begin{matrix} 0 & 0 & \cdots \\ \vdots & \vdots & \\ 0 & 0 & \cdots \\ 1 & 0 & \cdots \\ \vdots & \ddots & \\ 0 & \cdots & 1 \end{matrix}}_{\frac{p-1}{2}} & \begin{matrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} \end{matrix} \right|, \; C_{2n}^{(p)}[\beta|\alpha] = \left| \begin{matrix} \underbrace{\begin{matrix} 0 & 0 & \cdots \\ \vdots & \vdots & \\ 0 & 0 & \cdots \\ 1 & 0 & \cdots \\ \vdots & \ddots & \\ 0 & \cdots & 1 \end{matrix}}_{\frac{p+1}{2}} & \begin{matrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} \end{matrix} \right|$$

Lemmas, Propositions, Theorems

The product of the minors above is given by the following expressions

$$C_{2n}^{(p)}[\alpha|\beta]C_{2n}^{(p)}[\beta|\alpha] = (-1)^{(n-\frac{p-1}{2})\frac{p-1}{2}}(-1)^{(n-\frac{p+1}{2})\frac{p+1}{2}} = (-1)^{np-\frac{2p^2+2}{4}}$$

Since $p = 2k+1$, we have

$$np - \frac{2p^2+2}{4} = n(2k+1) - \frac{2(2k+1)^2+2}{4} = 2(nk-k^2-k)+n-1$$

And so,
the *basic* $p-$*circulant* permutation matrix $C_{2n}^{(p)}$ (p odd and $g.c.d.(p,n)=1$) is sign symmetric if n is odd and anti sign symmetric if n is even.

Lemmas, Propositions, Theorems

The case when $g.c.d.(p, n) \neq 1$ is more complicated.
After some Lemmas we can prove the following Theorem.

Theorem:

Let p be a positive integer, $C_{2n}^{(p)}$ the basic $p-$circulant permutation matrix, with $g.c.d.(p, n) = l$, and α_i $i = 1(1)2l$, different nonempty subsets of $\{1, 2, \ldots, 2n\}$ of cardinality $l_n = \frac{n}{l}$. Then the product $C_{2n}^{(p)}[\alpha|\beta]C_{2n}^{(p)}[\beta|\alpha] \neq 0$ and the order of determinants is minimal, if and only if

$$1)\ l_p \text{ is odd and } \{\alpha, \beta\} = \{\alpha_i, \alpha_{i+l}\}$$

$$2)\ l_p \text{ is even and } \{\alpha, \beta\} = \{\alpha_i, \alpha_i\}$$

where $l_p = \frac{p}{l}$.

The case when l_p is even is trivial and the matrix is *sign symmetric*.
In case l_p is odd, we call, for convenience,
the determinant

$$C_{2n}^{(p)}[\alpha|\beta], \text{ with } \alpha \in \{\alpha_i, i = 1(1)l\} \text{ and } \beta = \alpha_{i+l},$$

a determinant of type I and
the determinant

$$C_{2n}^{(p)}[\beta|\alpha], \text{ with } \beta \in \{\alpha_{i+l}, i = 1(1)l\} \text{ and } \alpha = \alpha_i,$$

a determinant of type II.
The following remarks can be readily checked.

- The two types of determinants, I and II, are determinants of basic p-circulant permutation matrices of order $l_n \times l_n$.
- The number of the two types of determinants is l.

- A determinant of type I has number 1 in the position $(1, 1+q_1)$, where q_1 is the largest integer less than $\frac{p-l}{2l}$.
- A determinant of type II has number 1 in the position $(1, 1+q_2)$, where q_2 is the largest integer less than $\frac{p}{2l}$.
- $q_2 = q_1 + 1$, since $\frac{p}{2l} - \frac{p-l}{2l} = \frac{1}{2}$.
- The union sets of α_i and the corresponding of α_{i+l} give determinants of the same type and analogous size. The total number of determinants of type I and type II is

$$\binom{l}{1} + \binom{l}{2} + \cdots + \binom{l}{l} = 2^l - 1$$

We can compute easily the type I and II determinants. So, we have

$$D_I = (-1)^{(l_n - q_1)q_1} \text{ and } D_{II} = (-1)^{(l_n - q_2)q_2}$$

In the same way we can compute determinants of type I and type II with $\alpha = \alpha_i \cup \alpha_j$, and $\beta = \alpha_{i+l} \cup \alpha_{j+l}$ $1 \leq i, j \leq l$ and we have

$$D_I = (-1)^{(2l_n - 2q_1)2q_1} \text{ and } D_{II} = (-1)^{(2l_n - 2q_2)2q_2}.$$

Finally it is easy to prove that the union of odd α_i's or even α_i's gives analogous results as previous ones.

Theorem:

Let p, n be positive integers, with $g.c.d.(p,n) = l \neq 1$, $l_p = \frac{p}{l}$, $l_n = \frac{n}{l}$, $C_{2n}^{(p)}$ the basic $p-$circulant permutation matrix, then

1) $l_p = even$. The matrix $C_{2n}^{(p)}$ is sign symmetric.
2) $l_p = odd$.
    - i) $l_n = odd$. The matrix $C_{2n}^{(p)}$ is sign symmetric.
    - ii) $l_n = even$. The matrix $C_{2n}^{(p)}$ is neither sign symmetric nor anti sign symmetric.

Hershkowitz and Keller proved that the matrix

$$A = \begin{pmatrix} x_1 & y_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & y_{n-1} \\ y_n & 0 & \ldots & 0 & x_n \end{pmatrix}$$

is neither sign symmetric nor anti sign symmetric, when the x_i's share the same sign, $\prod_{i=1}^{n} y_i > 0$ and n is even.

However, this is not true in a more general case. E.g., let the matrix

$$A_{4,2} = \begin{pmatrix} x_1 & 0 & y_1 & 0 \\ 0 & x_2 & 0 & y_2 \\ y_3 & 0 & x_3 & 0 \\ 0 & y_4 & 0 & x_4 \end{pmatrix},$$

then

Theorem:

Let x_i and y_i, $i = 1(1)4$, be real numbers. Then the matrix $A_{4,2}$ is sign symmetric if and only if $y_1 y_3 \geq 0$ and $y_2 y_4 \geq 0$. In all the other cases the matrix is neither sign symmetric nor anti sign symmetric.

Moreover, since a symmetric matrix is a sign symmetric one, then

Theorem:

The symmetric matrix $A_{2k,k}$, is a sign symmetric one.

An analogous of Hershkowitz and Keller theorem is valid in case where the order of matrix is odd. So, we have

Theorem:

Let $n > 2$ be an integer and $x_i, y_i, i = 1(1)2n+1$, be nonzero real numbers so that all x_i's share the same sign and $\prod_{i=1}^{2n+1} y_i > 0$. Then the matrix

$$A_{2n+1,2} = \begin{pmatrix} x_1 & 0 & y_1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ y_{2n} & & \ddots & x_{2n} & 0 \\ 0 & y_{2n+1} & \cdots & 0 & x_{2n+1} \end{pmatrix}$$

is neither sign symmetric nor anti sign symmetric.

Definition

A square real matrix A is called a P^S-matrix if A^k is a $P-$matrix for all $k \in S$, where S is a finite or an infinite set of positive numbers.

For convenience, we use the notation P^2 for the $P^{\{1,2\}}-$matrices.

Hershkowitz and Keller ask if P^2-matrices are stable. An answer to this question is that

There is a class of $A_{2n,2}$ matrices, such that if these are P^2-matrices then these are stable.

The $A_{2n,2}$ shifted circulant matrix has the form

$$A_{2n,2} = \begin{pmatrix} x & 0 & y & 0 & 0 & 0 \\ 0 & x & 0 & y & 0 & 0 \\ \vdots & & \ddots & & \ddots & \vdots \\ 0 & 0 & 0 & x & 0 & y \\ y & 0 & 0 & 0 & x & 0 \\ 0 & y & 0 & 0 & 0 & x \end{pmatrix}$$

We can prove that

Theorem:

Let $A_{2n,2}$ be a shifted circulant matrix, with $x, y \in \boldsymbol{R}$. This matrix is a $P-$matrix if and only if:

(i) $x > 0, x + y > 0$, *if n odd*

(ii) $x > 0, x^2 - y^2 > 0$, *if n even.*

Now the $A^2_{2n,2}$ matrix is the following

$$A^2_{2n,2} = \begin{pmatrix} x^2 & 0 & 2xy & 0 & y^2 & 0 & \cdots & 0 \\ 0 & x^2 & 0 & 2xy & 0 & y^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \ddots & \ddots & y^2 \\ y^2 & 0 & 0 & 0 & \ddots & 0 & \ddots & 0 \\ 0 & y^2 & 0 & 0 & 0 & \ddots & \vdots & 2xy \\ 2xy & 0 & y^2 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 2xy & 0 & y^2 & 0 & 0 & 0 & x^2 \end{pmatrix}$$

After some Lemmas and some simple Graph Theory we can prove that

Theorem:

Let $A_{2n,2}$ be a shifted circulant matrix, with $x, y \in \mathbb{R}$. If this matrix is a P^2-matrix, then $(x, y) \in \{(x, y) : x > 0 \wedge x^2 - y^2 > 0\}$.

The following lemma for circulant matrices is well known.

Lemma

Let ρ_i be the i^{th} of the n roots of unity. The eigenvalues of a circulant matrix are given by

$$\lambda_i = \sum_{k=1}^{n} a_k \rho_i^{k-1}, \quad i = 1(1)n$$

From this Lemma we have that the eigenvalues of $A_{2n,2}$ are

$$\lambda_l = x + ye^{i\frac{2(l-1)\pi}{2n}2} = x + y\cos\left(\frac{2(l-1)\pi}{n}\right) + iy\sin\left(\frac{2(l-1)\pi}{n}\right), \quad l = 1(1)2n$$

Apparently, if $x + y\cos(\frac{2(l-1)\pi}{n}) > 0$, the matrix $A_{2n,2}$ is stable. However, this is valid when the matrix $A_{2n,2}$ is a P^2-matrix.

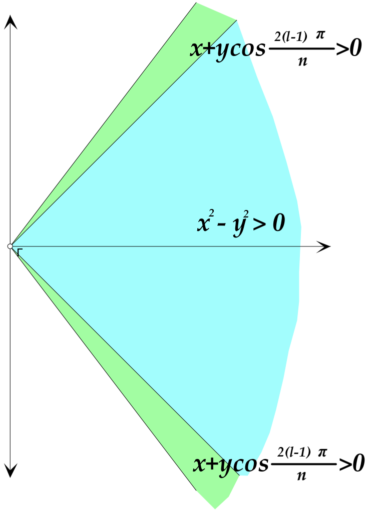

Figure: The regions

Let the shifted circulant matrix

$$A_{6,2} = \begin{pmatrix} x & 0 & y & 0 & 0 & 0 \\ 0 & x & 0 & y & 0 & 0 \\ 0 & 0 & x & 0 & y & 0 \\ 0 & 0 & 0 & x & 0 & y \\ y & 0 & 0 & 0 & x & 0 \\ 0 & y & 0 & 0 & 0 & x \end{pmatrix}$$

We denote $D_{|\alpha|} = A_{6,2}[\alpha|\beta]A_{6,2}[\beta|\alpha]$, where $\alpha, \beta \subset \{1,2,3,4,5,6\}$, with $|\alpha| = |\beta|$. We have $n_{|\alpha|} = \binom{6}{|\alpha|}$ sets α and $\binom{n_{|\alpha|}}{2}$ products $D_{|\alpha|}$. So, there exist $\sum_{|\alpha|=1}^{6} \binom{n_{|\alpha|}}{2} = 430$ products of the form $A_{6,2}[\alpha|\beta]A_{6,2}[\beta|\alpha]$, with $\alpha \neq \beta$.

From these products, 66 are different from zero and are distributed as follows:

- There are 6 products, $D_5 \neq 0$, of the form
$$D_5 = -xy(x+y)^2(x^2 - xy + y^2)^2 y^2$$
- There are 36 products, $D_4 \neq 0$, of the forms
$$D_4 = \begin{cases} -x^3y^5, & \text{(18 cases)} \\ \text{or} & \\ x^4y^6, & \text{(18 cases)} \end{cases}$$
- There are 18 products, $D_3 \neq 0$, of the form
$$D_3 = -x^3y^3$$
- There are 6 products, $D_2 \neq 0$, of the form
$$D_2 = -xy^3$$

So, we can state the following Theorems.

Theorem:

Let the shifted circulant matrix $A_{6,2}$. This matrix is sign symmetric if and only if $xy < 0$.

Since,
1) the spectrum of $A_{6,2}$ is given by

$$\sigma(A_{6,2}) = \{x + y, x - \frac{1}{2}y + i\frac{\sqrt{3}}{2}y, x - \frac{1}{2}y - i\frac{\sqrt{3}}{2}y\}$$

and

2)if A is a sign symmetric $n \times n$ matrix, then the next equivalence is valid :

The matrix A is stable. $\Leftrightarrow$ The matrix A is a $P-$matrix we can prove that

Theorem:

Let $A_{6,2}$ be a sign symmetric shifted circulant matrix. Then

(i) $x > 0$

$$A_{6,2} \text{ is a } P-matrix \Leftrightarrow x + y > 0$$

(ii) $x < 0 \Rightarrow A_{6,2}$ *is not a* $P-$*matrix*

Finally,

Theorem:

Let $A_{6,2}$ be a shifted circulant matrix, with $x, y \in \mathbb{R}$. This matrix is a P^2-matrix if and only if $x > 0, x + y > 0, x - y\sqrt[3]{2} > 0$.