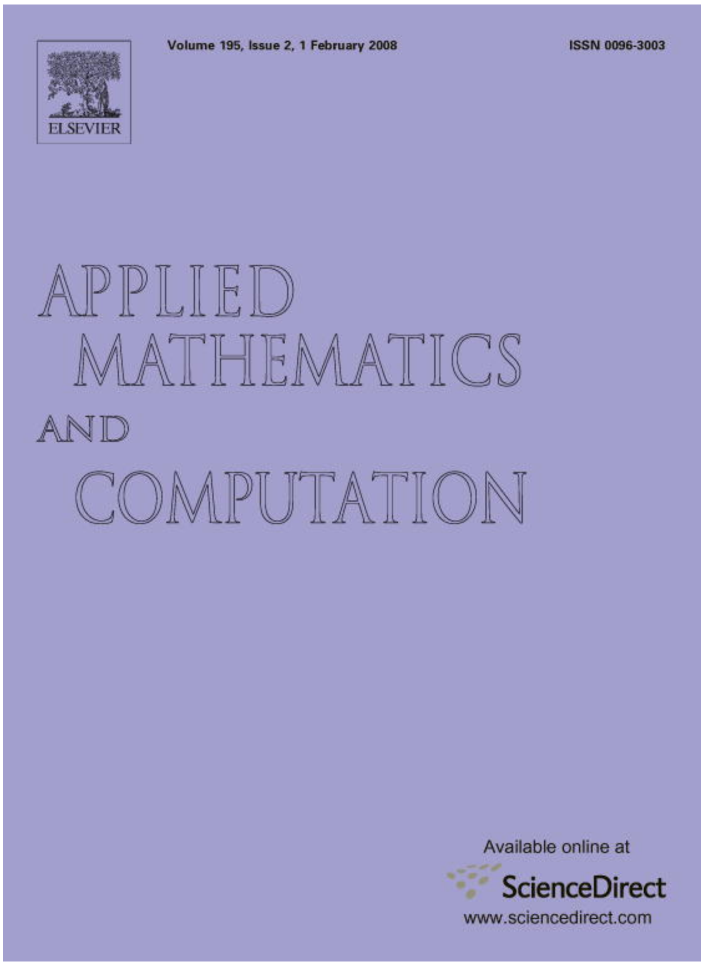
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On sign symmetric circulant matrices

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Abstract

In the last four decades many researchers have studied and analyzed the study of sign symmetry and positivity of principal minors of matrices, since these issues are related to stability. In this work we extend the theory about sign symmetric basic p-circulant permutation and sifted p-circulant matrices. We present and prove sufficient and necessary conditions for P-matrices and necessary conditions for P^2-matrices. Finally we present a class of matrices, where the P^2-matrices are stable.

Keywords: Sign symmetric matrices; Circulant matrices Q- and P-matrices; Positive stable matrices

1. Introduction and preliminaries

Consider two subsets α and β of $\{1, 2, \ldots, n\}$ with the same cardinality ($|\alpha| = |\beta|$) and an $n \times n$ square real matrix A. We denote by $A[\alpha|\beta]$ the minor with the rows indexed by α and columns indexed by β. If $\alpha = \beta$ the minor is a *principal* minor of A. The matrix A is called *sign symmetric* if $A[\alpha|\beta]A[\beta|\alpha] \geqslant 0$, for all α and $\beta \subset \{1, 2, \ldots, n\}$ with $|\alpha| = |\beta|$. The matrix A is called *weakly sign symmetric* if $A[\alpha|\beta]A[\beta|\alpha] \geqslant 0$, for all α and $\beta \subset \{1, 2, \ldots, n\}$ with $|\alpha| = |\beta| = |\alpha \cap \beta| + 1$.

A square real matrix A is called a *Q-matrix* (*Q_0-matrix*) if the sums of principal minors of A of the same order are positive (nonnegative), or equivalently a Q-matrix can be defined as the matrix whose characteristic polynomial has coefficients with alternating signs. A square real matrix A is called a *P-matrix* (*P_0-matrix*) if all the principal minors of A are positive (nonnegative). The positive definite matrices and the M-matrices belong to the class of P-matrices. The class of P-matrices satisfies properties $(A1)$–$(A6)$ of Theorem 6.2.3 in [5].

A square real matrix A is called a *P^S-matrix* (*Q^S-matrix*) if A^k is a *P-matrix* (*Q-matrix*) for all $k \in S$, where S is a finite or a infinite set of positive numbers. Hershkowitz and Keller [6] use the notation P^2 for the $P^{\{1,2\}}$-matrices and the same notation is adopted in this work.

Finally, a square real matrix A is called *positive stable* or simply *stable* if its eigenvalues have positive real parts or equivalently if its eigenvalues lie in the open right half complex-plane. A square real matrix A is called *semistable* if its eigenvalues have nonnegative real parts. For the important role of stability in applications the reader is referred to Refs. [3,4].

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Many researchers have studied the connection among the class of P-matrices with stability and sign symmetry (see, e.g., [1,2]). Recently Hershkowitz and Keller [7] have studied the sign symmetry of basic and shift basic circulant permutation matrices and have given a simple criterion for [anti] symmetric matrices of this class. They have dealt with 3×3 sign symmetric matrices and the arguments of their complex eigenvalues. The same researchers, in [6], studied the relation between positivity of principal minors, sign symmetry and stability of matrices. They devoted a large part of their work to discuss the relation between P^S-and Q^S-matrices and the sign symmetry. A number of open questions were raised in the aforementioned works and some answers are given in this paper.

Our work is organized as follows: In Section 2, we generalize Hershkowitz and Keller 's theory on sign symmetry of Circulant Permutation Matrices by presenting and proving more general statements. In Section 3, we study Shifted Circulant Permutation Matrices making clear that the results in [7] are not valid in the general case. In Section 4, we give a class of matrices, where a question raised in [6] is answered affirmatively. Finally, in Section 5, we give an example to confirm the theory developed.

2. Sign symmetry of circulant permutation matrices

Definition 2.1. A $n \times n$ real matrix is called a *circulant* matrix if it is of the form

$$C_n = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}. \tag{2.1}$$

The following lemma for circulant matrices is well known [8].

Lemma 2.1. *Let ρ_i be the ith of the n roots of unity. The eigenvalues of the circulant matrix (2.1) are given by*

$$\lambda_i = \sum_{k=1}^{n} a_k \rho_i^{k-1}, \quad i = 1(1)n^1. \tag{2.2}$$

Definition 2.2. An $n \times n$ matrix is called a *basic p*-circulant permutation matrix if it is defined as follows

$$(C_n^{(p)})_{ij} = \begin{cases} 1 \; j = i + p & \text{if } 1 \leqslant i \leqslant n - p, \\ 1 \; j = i - n + p & \text{if } n - p < i \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

The *basic p*-circulant permutation matrix has the form

$$C_n^{(p)} = \begin{pmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & & \vdots \\ \underbrace{0 \quad \cdots \quad 1}_{p} & & & 0 & \cdots & 0 \end{pmatrix}.$$

[1] The notation a(b)c is an abbreviation of all the terms of the arithmetic progression with first term a, step b>0(<0) and last term the largest (smallest) one that is not greater (smaller) than c.

Proposition 2.1. *Let p,n be positive integers with $g.c.d.(p,n) = 1$. Then there holds*

$$\{x \in \mathbb{N} : x \equiv kp \pmod{n},\ k = 1(1)n\} = \{0, 1, \ldots, n-1\}. \tag{2.3}$$

Proof. Let $A = \{x \in \mathbb{N} : x \equiv kp \pmod{n}, k = 1(1)n\}$. If $x_1, x_2 \in A$, then $x_1 \neq x_2$, since $x_1 = x_2 \Rightarrow k_1 p \equiv k_2 p \pmod{n}$ with $k_1, k_2 \leqslant n \Rightarrow (k_1 - k_2)p \equiv 0 \pmod{n}$. But this means that the $l.c.m.(p,n) < pn$, which contradicts the assumption $g.c.d.(p,n) = 1$. $\square$

Theorem 2.1. *Let p be a positive integer, $C_{2n}^{(p)}$ the basic p-circulant permutation matrix, with $g.c.d.(p,n) = 1$, and α, β different nonempty subsets of $\{1, 2, \ldots, 2n\}$ of the same cardinality. The product $C_{2n}^{(p)}[\alpha|\beta]C_{2n}^{(p)}[\beta|\alpha] \neq 0$ if and only if*

$$\{\alpha, \beta\} = \{\{1, 3, \ldots, 2n-1\}, \{2, 4, \ldots, 2n\}\}. \tag{2.4}$$

Proof. It is clear that

$$C_{2n}^{(p)}[\alpha|\beta]C_{2n}^{(p)}[\beta|\alpha] \neq 0 \iff C_{2n}^{(p)}[\alpha|\beta] \neq 0 \quad \text{and} \quad C_{2n}^{(p)}[\beta|\alpha] \neq 0. \tag{2.5}$$

We note that if $i \notin \alpha$ and $i + p \in \beta$ (where $i + p$ is identified with $i + p \pmod{2n}$), then $C_{2n}^{(p)}[\alpha|\beta] = 0$, since the column $i + p \pmod{2n}$ contains only zeros. The first term of the right part of (2.5) implies that if $i \notin \alpha \Rightarrow i + p \in /\beta$, which, by the second term of (2.5) implies that $i + 2p \notin \alpha$. In general, for $k = 1, 2, \ldots,$ we have

$$i \notin \alpha \Rightarrow i + 2kp \pmod{2n} \notin \alpha, \tag{2.6}$$
$$i \notin \beta \Rightarrow i + 2kp \pmod{2n} \notin \beta, \tag{2.7}$$

where $0 \pmod{2n}$ is taken as $2n$.

From Proposition 2.1, it follows that:

$$\{x \in \mathbb{N} : x \equiv 2kp \pmod{2n}, k = 1(1)n\} = \{2, 4, \ldots, 2n\}, \text{ while}$$
$$\{x \in \mathbb{N} : x \equiv 2kp + 1 \pmod{2n}, k = 1(1)n\} = \{1, 3, \ldots, 2n-1\}.$$

So, if we take $i = 1$ then from (2.6) $\beta = \{1, 3, \ldots, 2n-1\}$, whereas if we take $i = 2$ then from (2.7) $\alpha = \{2, 4, \ldots, 2n\}$.

Reversely, let p be even. In this case, since $\alpha \neq \beta$ and the number $+1$ is located in positions with only odd or even indices, we have $C_{2n}^{(p)}[\alpha|\beta] = 0$. Also, $\alpha = \beta \Rightarrow C_{2n}^{(p)}[\alpha|\beta] = 1$ and so the matrix $C_{2n}^{(p)}$ is sign symmetric.

Let p be odd. In this case the minors have the form:

$$C_{2n}^{(p)}[\alpha|\beta] = \begin{vmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & & \vdots \\ \underbrace{0 \quad \cdots \quad 1}_{\frac{p-1}{2}} & & & 0 & \cdots & 0 \end{vmatrix}, \quad C_{2n}^{(p)}[\beta|\alpha] = \begin{vmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & & \vdots \\ \underbrace{0 \quad \cdots \quad 1}_{\frac{p+1}{2}} & & & 0 & \cdots & 0 \end{vmatrix}$$

The product of the minors above is given by the following expressions:

$$C_{2n}^{(p)}[\alpha|\beta]C_{2n}^{(p)}[\beta|\alpha] = (-1)^{\left(n-\frac{p-1}{2}\right)\frac{p-1}{2}}(-1)^{\left(n-\frac{p+1}{2}\right)\frac{p+1}{2}} = (-1)^{np-\frac{2p^2+2}{4}}. \qquad \square \tag{2.8}$$

Corollary 2.1. *The basic p-circulant permutation matrix $C_{2n}^{(p)}$ (p odd and $g.c.d.(p,n) = 1$) is sign symmetric if n is odd and anti sign symmetric if n is even.*

Proof. Since $p = 2k + 1$, it is obvious that the exponent in (2.8) is given by

$$np - \frac{2p^2 + 2}{4} = n(2k+1) - \frac{2(2k+1)^2 + 2}{4} = 2(nk - k^2 - k) + n - 1. \qquad \square$$

The case where $g.c.d.(p, n) \neq 1$ is more complicated. We begin our analysis by giving the following lemma.

Lemma 2.2. *Let p, n be two positive integers with $g.c.d.(p, n) = l \neq 1$. There are $2l$ classes (of integers) of cardinality $\frac{n}{l}$:*

$$a_i = \left\{x \in \mathbb{N} : x = 2kp + i \pmod{2n}, k = 1(1)\frac{n}{l}\right\} = \{i, i + 2l, i + 4l, \ldots, i + 2(n - l)\}, \quad i = 1(1)2l.$$

Proof. Let $l_p = \frac{p}{l}$ and $l_n = \frac{n}{l}$, then $g.c.d.(l_p, l_n) = 1$. Proposition (2.1) implies that

$$a_{2l_n} := \{x \in \mathbb{N} : x = kl_p \pmod{l_n}, k = 1(1)l_n\} = \{1, 2, \ldots, l_n\}, \tag{2.9}$$

where $0 \pmod{l_n}$ is taken as l_n, or equivalently

$$\begin{aligned} a_{2l} &:= \{x \in \mathbb{N} : x = 2kl_pl \pmod{2l_nl}, k = 1(1)l_n\} = \{x \in \mathbb{N} : x = 2kp \pmod{2n}, k = 1(1)l_n\} \\ &= \{2l, 4l, \ldots, 2n\}. \end{aligned} \tag{2.10}$$

Similarly, we can give the other sets a_i.

Moreover, since

$$a_i \cap a_j = \varnothing \quad \text{and} \quad \bigcup_i a_i = \{0, 1, \ldots, 2n - 1\}$$

the statement is true. $\square$

Lemma 2.3. *Let p, n be two positive integers with $g.c.d.(p, n) = l \neq 1$, a_i, $i = 1(1)2l$, be the classes of the previous lemma and $l_p = \frac{p}{l}$. Then*

$$a_{i+p} = a_k, \quad \text{where } k = \begin{cases} i + l \pmod{2l} & \text{if } l_p \text{ odd}, \\ i \pmod{2l} & \text{if } l_p \text{ even}. \end{cases}$$

Proof. The above is obvious, by Lemma 2.2, since $p = l_pl$ and

$$i + p = i + l_pl = \begin{cases} i + l + 2kl & \text{if } l_p \text{ odd}, \\ i + 2kl & \text{if } l_p \text{ even}. \end{cases} \qquad \square$$

Theorem 2.2. *Let p be a positive integer, $C_{2n}^{(p)}$ the basic p-circulant permutation matrix, with $g.c.d.(p, n) = l$, and α_i $i = 1(1)2l$, different nonempty subsets of $\{1, 2, \ldots, 2n\}$ of cardinality $l_n = \frac{n}{l}$. The product $C_{2n}^{(p)}[\alpha|\beta]C_{2n}^{(p)}[\beta|\alpha] \neq 0$ and the order of determinants is minimal, if and only if*

$$l_p \text{ is odd and } \{\alpha, \beta\} = \{\alpha_i, \alpha_{i+l}\} \text{ or } l_p \text{ is even and } \{\alpha, \beta\} = \{\alpha_i, \alpha_i\}, \tag{2.11}$$

where $l_p = \frac{p}{l}$.

Proof. Here, relationship (2.5) holds. The fact that $C_{2n}^{(p)}[\alpha|\beta] \neq 0$ implies that there exists an element $a_{i,i+p} = 1$ in each row and in each column of $C_{2n}^{(p)}[\alpha|\beta]$. Now, if $i \in \alpha$, then $i + p \in \beta$ and consequently $i + 2p \in \alpha$. The fact that $C_{2n}^{(p)}[\beta|\alpha] \neq 0$ implies that there exists an element $a_{i+p,i+2p} = 1$ in each row and in each column of $C_{2n}^{(p)}[\beta|\alpha]$. Therefore, if $i + p \in \beta$, then $i + 2p \in \alpha$ and so $i + 3p \in \beta$. This means, by Lemma 2.3, that

$$\alpha = \{x \in \mathbb{N} : x = i + 2kp \pmod{2n}\} \in \{\alpha_i, i = 1(1)2l\} \quad \text{and} \quad \beta = \alpha_{i+l}$$

in case $l_p = \text{odd}$, while if $l_p = \text{even}$ then $\alpha = \beta \in \{\alpha_i, i = 1(1)2l\}$. $\square$

The case where $l_p = \text{even}$ is trivial and the matrix is *sign symmetric*. In case l_p is odd, we call, for convenience, the determinant $C_{2n}^{(p)}[\alpha|\beta]$, with $\alpha \in \{\alpha_i, i = 1(1)l\}$ and $\beta = \alpha_{i+l}$, a determinant of type I and the determinant $C_{2n}^{(p)}[\beta|\alpha]$, with $\beta \in \{\alpha_{i+l}, i = 1(1)l\}$ and $\alpha = \alpha_i$, a determinant of type II.

The following remarks can be readily checked.

- The two types of determinants, I and II, are determinants of basic p-circulant permutation matrices of order $l_n \times l_n$.
- The number of the two types of determinants is l.
- A determinant of type I has a $+1$ in the position $(1, 1 + q_1)$, where q_1 is the largest integer less than $\frac{p-l}{2l}$, since there must hold $(2k + 1)l < p$.
- A determinant of type II has a $+1$ in the position $(1, 1 + q_2)$, where q_2 is the largest integer less than $\frac{p}{2l}$, since there must hold $2kl < p$.
- $q_2 = q_1 + 1$, since $\frac{p}{2l} - \frac{p-l}{2l} = \frac{1}{2}$ (recall that $p = l_p l$ and l_p is odd).
- The union sets of α_i and the corresponding of α_{i+l} give determinants of the same type and analogous size. The total number of determinants of type I and type II is

$$\binom{l}{1} + \binom{l}{2} + \cdots + \binom{l}{l} = 2^l - 1.$$

We can compute a type I determinant by moving the q_1 bottom rows to the top and, similarly, a type II determinant by moving the q_2 bottom rows to the top. So, we have

$$D_{\mathrm{I}} = (-1)^{(l_n - q_1)q_1} \quad \text{and} \quad D_{\mathrm{II}} = (-1)^{(l_n - q_2)q_2}. \tag{2.12}$$

From (2.12) we find

$$C_{2n}^{(p)}[\alpha|\beta]C_{2n}^{(p)}[\beta|\alpha] = D_{\mathrm{I}}D_{\mathrm{II}} = (-1)^{(l_n - q_1)q_1 + (l_n - q_2)q_2} = (-1)^{l_n - 1}. \tag{2.13}$$

In the same way we can compute determinants of type I and type II with $\alpha = \alpha_i \cup \alpha_j, 1 \leqslant i, j \leqslant l$ and $\beta = \alpha_{i+l} \cup \alpha_{j+l}$. For this we have

$$D_{\mathrm{I}} = (-1)^{(2l_n - 2q_1)2q_1} \quad \text{and} \quad D_{\mathrm{II}} = (-1)^{(2l_n - 2q_2)2q_2} \tag{2.14}$$

and

$$C_{2n}^{(p)}[\alpha|\beta]C_{2n}^{(p)}[\beta|\alpha] = D_{\mathrm{I}}D_{\mathrm{II}} = 1. \tag{2.15}$$

It is easy to prove that the union of odd α_i's gives a similar result to that in relation (2.13) while the union of even α_i's gives a result as in the relation (2.15). After the above analysis we introduce the next theorem.

Theorem 2.3. *Let p, n be positive integers, with g.c.d.$(p, n) = l \neq 1$, $l_p = \frac{p}{l}$, $l_n = \frac{n}{l}$, $C_{2n}^{(p)}$ the basic p-circulant permutation matrix, then*

(1) *$l_p = \text{even}$. The matrix $C_{2n}^{(p)}$ is sign symmetric.*
(2) *$l_p = \text{odd}$.*
  (i) *$l_n = \text{odd}$. The matrix $C_{2n}^{(p)}$ is sign symmetric.*
  (ii) *$l_n = \text{even}$. The matrix $C_{2n}^{(p)}$ is not sign symmetric nor anti sign symmetric.*

3. On shifted circulant permutation matrices

Hershkowitz and Keller [7] proved that the matrix

$$A = \begin{pmatrix} x_1 & y_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & y_{n-1} \\ y_n & 0 & \ldots & 0 & x_n \end{pmatrix}, \tag{3.1}$$

where the x_i's share the same sign and $\prod_{i=1}^{n} y_i > 0$, in case n is even, is neither sign symmetric nor anti sign symmetric. However, this is not true in a more general case. For example, let the matrix

$$A_{4,2} = \begin{pmatrix} x_1 & 0 & y_1 & 0 \\ 0 & x_2 & 0 & y_2 \\ y_3 & 0 & x_3 & 0 \\ 0 & y_4 & 0 & x_4 \end{pmatrix}.$$

Since $a_{ij}a_{ji} \neq 0, i = 1(1)4$, and $j = i + 2 \pmod 4$, it is easy to verify that

(i) When α and β are subsets of $\{1, 2, 3, 4\}$ of cardinality +1

$$A_{4,2}[\alpha|\beta]A_{4,2}[\beta|\alpha] = y_{i+1}y_{i+3}, \quad i = 0, 1.$$

(ii) When α and β are subsets of $\{1, 2, 3, 4\}$ of cardinality +2

$$A_{4,2}[\alpha|\beta]A_{4,2}[\beta|\alpha] = \begin{cases} x_i^2 y_{i+1}y_{i+3} \\ \text{or} \\ y_1y_2y_3y_4 \end{cases}, \quad i = 1(1)4.$$

(iii) When α and β are subsets of $\{1, 2, 3, 4\}$ of cardinality +3

$$A_{4,2}[\alpha|\beta]A_{4,2}[\beta|\alpha] = y_{i+1}y_{i+3}(x_{i+2}x_{i+4} - y_{i+2}y_{i+4})^2, \quad i = 0, 1.$$

Note: Recall that all indices are mod 4 and $\star_0 = \star_4$.

So, since in all other cases there holds $A[\alpha|\beta]A[\beta|\alpha] = 0$, the next theorem is valid.

Theorem 3.1. *Let x_i and y_i, $i = 1(1)4$, be real numbers. Then the matrix $A_{4,2}$ is sign symmetric if and only if $y_1y_3 \geqslant 0$ and $y_2y_4 \geqslant 0$. In all other cases the matrix is neither sign symmetric nor anti sign symmetric.*

We consider now a more general form than that in (3.1) for shifted circulant matrices

$$A_{n,k} = \begin{pmatrix} x_1 & 0 & \dots & 0 & y_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 & 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & & \dots & x_{n-k} & 0 & \dots & \dots & y_{n-k} \\ y_{n-k+1} & 0 & \dots & 0 & x_{n-k+1} & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & y_n & 0 & \dots & \dots & x_n \end{pmatrix}.$$

Since a symmetric matrix is a sign symmetric one, then

Theorem 3.2. *The matrix $A_{2k,k}$ with $x_i = x$ and $y_i = y$, $i = 1(1)2k$, is a sign symmetric one.*

Lemma 3.1. *Let x, y be real numbers and a matrix $B_n = (b_{ij})$, where*

$$b_{ij} = \begin{cases} x, & j = i + 1, \quad i = 1(1)n - 1, \\ y, & j = i - 1, \quad i = 2(1)n, \\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

Then

$$\det(B_n) = -xy\det(B_{n-2}). \tag{3.3}$$

Moreover $det(B_{2n}) = (-1)^n x^n y^n$ and $det(B_{2n+1}) = 0$.

Proof. Relation (3.3) is obvious if we expand $\det(B_n)$ twice in the terms of its first row. The other relations are easy to prove, since $\det(B_2) = -xy$ and $\det(B_3) = 0$. □

Theorem 3.3. *Let x, y be nonzero real numbers and $n > 1$ a positive integer. Then the shifted circulant permutation matrices $A_{2n+1,2}$ is neither sign symmetric nor anti sign symmetric.*

Proof. We have

$$D_{2n} := A_{2n+1,2}[1,2,\ldots,2n-1,2n+1|1,2,\ldots,2n-1,2n] = \begin{vmatrix} x & 0 & y & 0 & \cdots & 0 \\ 0 & x & 0 & y & \cdots & 0 \\ & & & & \vdots & \\ 0 & y & 0 & 0 & \cdots & 0 \end{vmatrix}.$$

We expand the determinant once in the terms of its first column and once in the terms of its last row. Then, from Lemma 3.1 we have

$$D_{2n} = yx \det B_{2(n-1)} = xy(-1)^{n-1}x^{n-1}y^{n-1} = (-1)^{n-1}x^n y^n, \tag{3.4}$$

$$\widehat{D}_{2n} := A_{2n+1,2}[1,2,\ldots,2n-1,2n|1,2,\ldots,2n-1,2n+1] = \begin{vmatrix} x & 0 & y & 0 & \cdots & 0 & 0 \\ 0 & x & 0 & y & \cdots & 0 & 0 \\ & & & & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix}.$$

Now, we expand this determinant once in the terms of its last column and once in the terms of its last row. Then from Lemma 3.1 we have

$$\widehat{D}_{2n} = -y^2 \det B_{2(n-1)} = -y^2(-1)^{n-1}x^{n-1}y^{n-1} = (-1)^n x^{n-1}y^{n+1}. \tag{3.5}$$

From (3.4) and (3.5) we take

$$D_{2n}\widehat{D}_{2n} = -xy(x^{n-1}y^n)^2. \tag{3.6}$$

In an analogous way we find that

$$D_{2n-1} := A_{2n+1,2}[1,2,\ldots,2n-3,2n,2n+1|1,2,\ldots,2n-1] = y^{2n-1}$$

and

$$\widehat{D}_{2n-1} := A_{2n+1,2}[1,2,\ldots,2n-1|1,2,\ldots,2n-3,2n,2n+1] = y^2x^{2n-3}$$

and finally

$$D_{2n-1}\widehat{D}_{2n-1} = xy(x^{n-2}y^n)^2. \tag{3.7}$$

So, relations (3.6) and (3.7) prove the theorem. □

We are now able to prove a theorem analogous to Theorem 2.27 in [7].

Theorem 3.4. *Let $n > 2$ be an integer and $x_i, y_i, i = 1(1)2n+1$, be nonzero real numbers so that all x_i's share the same sign and $\prod_{i=1}^{2n+1} y_i > 0$. Then the matrix*

$$A_{2n+1,2} = \begin{pmatrix} x_1 & 0 & y_1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ y_{2n} & & \ddots & x_{2n} & 0 \\ 0 & y_{2n+1} & \cdots & 0 & x_{2n+1} \end{pmatrix}$$

is neither sign symmetric nor anti sign symmetric.

Proof. Without loss of generality we may assume that $x_1 = x_2 = \cdots = x_{2n+1} = x$ since sign and anti sign symmetry are invariant under multiplication of the matrix by a positive diagonal matrix. Moreover, we define $r = (y_1 y_2 \cdots y_{2n+1})^{1/(2n+1)}$ and the diagonal matrix $D = (d_1, d_2, \ldots, d_{2n+1})$, where

$$d_i = \begin{cases} \dfrac{r^{n-l+1}}{\prod_{j=l}^{n} y_{2j-1}}, & i = 2l-1, \\ \dfrac{r^{2n-l+1}}{\prod_{k=1}^{n} y_{2k-1} \prod_{j=l}^{n} y_{2j}}, & i = 2l, \qquad l = 1(1)n. \\ 1, & i = 2n+1, \end{cases}$$

Then, the matrix $D^{-1}A_{2n+1,2}D$ is a shifted circulant permutation matrices and by virtue of Theorem 3.3 our claim is proved. $\square$

4. On positivity of principal minors of a shift circulant matrix $A_{2n,2}$

Let $x, y \in IR$. The $A_{2n,2}$ shift circulant matrix has the form

$$A_{2n,2} = \begin{pmatrix} x & 0 & y & 0 & 0 & 0 \\ 0 & x & 0 & y & 0 & 0 \\ \vdots & & \ddots & & \ddots & \vdots \\ 0 & 0 & 0 & x & 0 & y \\ y & 0 & 0 & 0 & x & 0 \\ 0 & y & 0 & 0 & 0 & x \end{pmatrix}. \tag{4.1}$$

Theorem 4.1. *Let $A_{2n,2}$ be a shift circulant matrix, with $x, y \in IR$. This matrix is a P-matrix if and only if:*

(i) $x > 0, x + y > 0$, *if n odd.*
(ii) $x > 0, x^2 - y^2 > 0$, *if n even.*

Proof. The graph of $A_{2n,2}$ is of the form shown in Fig. 1. This means that there exists a permutation matrix P, so that the product $P^{-1}A_{2n,2}P$ will have a block diagonal form, where the diagonal elements are the same with shift circulant basic matrix C_n and where $\det(C_n) = x^n + (-1)^{n+1}y^n$. The graph of the matrix C_n is of type I as in Fig. 1. The type II, in Fig. 1, is the graph of a shift circulant basic matrix C_n with zero in the position $(n, 1)$, that is a matrix with elements in the main diagonal and in its first upper-diagonal.

A principal minor of a matrix results by removing some rows and the corresponding columns. In its graph, this means that we remove one or more nodes and it is clear that the new graph consists of sub-graphs of type II, in Fig. 1, and of at most one graph of type I, in the same figure. In matrix form, it means that there exists a permutation matrix P which transforms the principal matrix in a block diagonal matrix where its principal minor takes the value

$$\begin{aligned} &D_k = x^k \text{ or } D_k = x^j(x^n + (-1)^{n+1}y^n) && \text{if } k \geqslant n, \\ &D_k = x^k && \text{if } k < n. \end{aligned} \tag{4.2}$$

This relation with the fact that $D_1 = x$ and $D_{2n-1} = x^{n-1}(x^n + (-1)^{n+1}y^n)$ prove the theorem. $\square$

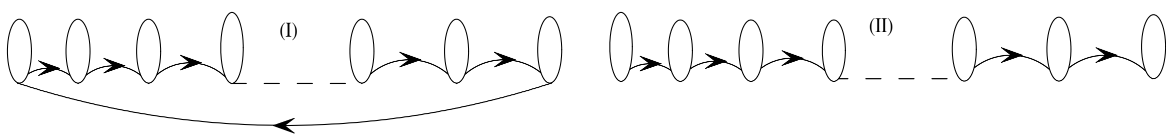

Fig. 1. The graph of the matrix C_n of type I and type II.

In the following we study the signs of principal minors of $A^2_{2n,2}$. For this we give some essential lemmas.

Lemma 4.1. *Let the $n \times n$ tridiagonal matrix, $A_n = (a_{ij})$, where*

$$a_{ij} = \begin{cases} x^2, & j = i-1, \ i = 2(1)n, \\ 2xy, & j = i, \ i = 1(1)n, \\ y^2, & j = i+1, \ i = 1(1)n-1. \end{cases}$$

Then

$$\det(A_n) = (n+1)x^n y^n. \tag{4.3}$$

Proof. Let $D_n = \det(A_n)$, then it is easy to check, by induction, that the relationship $D_n = 2xyD_{n-1} - x^2y^2D_{n-2}$ is valid and this implies (4.3). $\square$

Two yet matrices are important for our analysis.

Lemma 4.2. *Let the $n \times n$ matrix,*

$$B_n = \begin{pmatrix} x^2 & 2xy & y^2 & \cdots & 0 & 0 \\ 0 & x^2 & 2xy & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & \\ 0 & 0 & 0 & \ddots & 2xy & y^2 \\ y^2 & 0 & 0 & \ddots & x^2 & 2xy \\ 2xy & y^2 & 0 & \cdots & 0 & x^2 \end{pmatrix}. \tag{4.4}$$

Then

$$\det(B_n) = (x^n - (-1)^n y^n)^2. \tag{4.5}$$

Proof. We conveniently expand and we have

$$\begin{aligned} \det(B_n) &= \cdots = (x^2)^n + (-1)^n x^2y^2D_{n-2} + (y^2)^n + (-1)^n x^2y^2D_{n-2} + (-1)^n 2xyD_{n-1} = \cdots \\ &= x^{2n} + y^{2n} + (-1)^{n-1}2x^ny^n. \qquad \square \end{aligned}$$

The graph of the matrix B_n is presented in Fig. 2. The loops on the nodes have weight x^2, the paths P_iP_{i+1} have weight $2xy$ while the weight of the paths P_iP_{i+2} is y^2. We note that P_{n+1} is equivalent to P_1 and P_{n+2} is equivalent to P_2. $\square$

Lemma 4.3. *Let the $n \times n$ matrix,*

$$\widehat{B}_n = \begin{pmatrix} x^2 & 2xy & y^2 & \cdots & 0 \\ 0 & x^2 & 2xy & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & \ddots & 2xy \\ y^2 & 0 & 0 & \cdots & x^2 \end{pmatrix}. \tag{4.6}$$

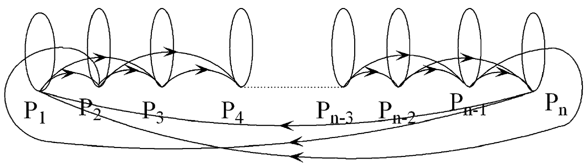

Fig. 2. The graph of the matrix B_n.

Then

$$\det(\widehat{B}_n) = x^{2n} - (-1)^n n x^{n-1} y^{n+1}. \tag{4.7}$$

Proof. It is easy to check. $\square$

Lemma 4.4. *Removing the kth, $k \neq 1, n$, row and column of the above matrix (4.6) results in a matrix $\widetilde{B}_n$ with x^2, y^2, x^2 in the $(k-1, k-1)$, $(k-1, k)$ and (k, k) positions, respectively. Moreover,*

$$\det(\widetilde{B}_n) = x^{2n} - (-1)^n y^2 (\det A_{k-2}) y^2 (\det A_{n-k}), \tag{4.8}$$

where the matrix A_n has a determinant given by (4.3).

Proof. Removing a row and a column, not the first or the last one, a matrix of the following block form arises:

$$\widetilde{B} = \begin{pmatrix} a_1 & \widehat{A}_1 & 0 & 0 \\ 0 & a_2^T & y^2 & 0 \\ 0 & 0 & a_3 & \widehat{A}_2 \\ y^2 & 0 & 0 & a_4^T \end{pmatrix}, \tag{4.9}$$

where a_1^T and a_3^T have the form $(x^2, 0, \ldots, 0)$, a_2^T and a_4^T the form $(0, \ldots, 0, x^2)$ while $\widehat{A}_1$ and $\widehat{A}_2$ are tridiagonal matrices with diagonal $2xy$ and sub-diagonals x^2 and y^2. Now, it is easy to check that the principal minor is $\det(\widetilde{B}_n) = x^{2n} - (-1)^n y^2 (\det A_{k-2}) y^2 (\det A_{n-k})$.

Let now the $A^2_{2n,2}$ matrix

$$A^2_{2n,2} = \begin{pmatrix} x^2 & 0 & 2xy & 0 & y^2 & 0 & \cdots & 0 \\ 0 & x^2 & 0 & 2xy & 0 & y^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \ddots & \ddots & y^2 \\ y^2 & 0 & 0 & 0 & \ddots & 0 & \ddots & 0 \\ 0 & y^2 & 0 & 0 & 0 & \ddots & \vdots & 2xy \\ 2xy & 0 & y^2 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 2xy & 0 & y^2 & 0 & 0 & 0 & x^2 \end{pmatrix}.$$

The graph of this matrix consists of two independent strongly connected subgraphs of the forms given in Fig. 2. By a permutation matrix P we can transform this matrix into a block diagonal matrix, where in the two diagonal blocks we have the matrix B_n of (4.4). So, we study the new block diagonal matrix.

A principal sub-matrix of this matrix can result by removing some rows and the corresponding columns. In its graph this means that we remove some nodes. If we remove nodes only from the one sub-graph, then in the principal minor of the matrix the factor $(x^n - (-1)^n y^n)^2$ (4.5) is present but obviously this does not change the sign of its determinant. So, the sign of a principal minor is changed only if we remove some nodes of a sub-graph. Now, we can distinguish two cases. In the first case, we can remove at least two consecutive nodes of the graph. Then, the strong connection of the graph is lost and with a permutation matrix P we can transform the sub-matrix into an upper triangular one, with diagonal elements x^2. So, in the principal minor of this matrix there is a factor x^{2k} which does not change the sign of it. In the second case we remove nodes, but these are not successive. Supposing that we remove only one row and the corresponding column. A permutation matrix P can transform this matrix into $\widehat{B}_{n-1}$ (4.6) and the principal minor is analogous to $x^{n-2}(x^n - (-1)^{n-1}(n-1)y^n)$. Removing more rows and columns (but not the first, the last or consecutive ones) a sub-matrix of analogous form as in (4.8) arises and so the principal minor is analogous to a term of the form $x^{k_1}(x^{k_2} \pm ky^{k_2}), k > 2$. □

The above analysis gives us the necessary background for the next theorem.

Theorem 4.2. *Let $A_{2n,2}$ be a shift circulant matrix, with $x, y \in IR$. If this matrix is a P^2-matrix, then $(x, y) \in \{(x, y) : x > 0 \wedge x^2 - y^2 > 0\}$.*

Proof. Let n be even. Since a P^2-matrix is a P-matrix, the validity of our claim is obvious by Theorem 4.1.

Let n be odd. The principal matrix of order $2n - 1$ is a matrix of a 2×2 block diagonal matrix, where the first diagonal block is the matrix (4.4) while the second diagonal block is the matrix (4.6). So, the principal minor of this is analogous to the term

$$x^{2(n-1)} - (-1)^{n-1}(n-1)x^{n-2}y^n = x^{n-2}(x^n - (n-1)y^n).$$

Since our $A_{2n,2}$ matrix is a P^2-matrix, the following relationship must be valid:

$$x^{n-2}(x^n - (n-1)y^n) > 0 \Longleftrightarrow y < \frac{1}{\sqrt[n]{n-1}}x$$

This relationship along with Theorem 4.1 prove the validity of the theorem too. □

Remark 4.1. The statement of Theorem 4.2 can be improved. We note that the proof of the theorem uses only minors of size $1, n-1$ and n, where n is the size of the matrix. So, in a circulant matrix all the minors of order $n-1$ are equal to each other, as well as all the minors of size $+1$. Hence, if the matrix is a Q-matrix, all its minors of size $1, n-1, n$ are positive, as in a P-matrix. Hence, the statement of the theorem can be improved by replacing the P^2-assumption by a weaker Q^2-assumption.

In [6], Question 6.2, Hershkowitz and Keller ask if P^2-matrices are stable. Below we prove, that the P^2-matrices $A_{2n,2}$ are stable.

From Lemma 2.1, the eigenvalues of $A_{2n,2}$ in (4.1) are

$$\lambda_l = x + ye^{i\frac{2(l-1)\pi}{2n}2} = x + ye^{i\frac{2(l-1)\pi}{n}}, \quad l = 1(1)2n. \tag{4.10}$$

It is obvious that all these n eigenvalues are of multiplicity two each. So, we have

$$\lambda_l = x + y\cos\left(\frac{2(l-1)\pi}{n}\right) + iy\sin\left(\frac{2(l-1)\pi}{n}\right), \quad l = 1(1)n. \tag{4.11}$$

Apparently, if $x + y\cos\left(\frac{2(l-1)\pi}{n}\right) > 0$ the matrix (4.1) is stable. However, from Theorem 4.2, this is valid when the matrix (4.1) is a P^2-matrix. Since, the answer to Hershkowitz and Keller's question is positive, for a class of matrices, we think this question should be restated as follows.

Question: Which classes of P^2-matrices are stable?

5. The shift circulant matrix $A_{6,2}$

Let the shift circulant matrix

$$A_{6,2} = \begin{pmatrix} x & 0 & y & 0 & 0 & 0 \\ 0 & x & 0 & y & 0 & 0 \\ 0 & 0 & x & 0 & y & 0 \\ 0 & 0 & 0 & x & 0 & y \\ y & 0 & 0 & 0 & x & 0 \\ 0 & y & 0 & 0 & 0 & x \end{pmatrix}. \tag{5.1}$$

We denote $D_{|\alpha|} = A_{6,2}[\alpha|\beta]A_{6,2}[\beta|\alpha]$, where $\alpha, \beta \subset \{1, 2, 3, 4, 5, 6\}$, with $|\alpha| = |\beta|$. We have $n_{|\alpha|} = \begin{pmatrix} 6 \\ |\alpha| \end{pmatrix}$ sets α and $\begin{pmatrix} n_{|\alpha|} \\ 2 \end{pmatrix}$ products $D_{|\alpha|}$. So, there exist $\sum_{|\alpha|=1}^{6} D_{|\alpha|} = 430$ products of the form $A_{6,2}[\alpha|\beta]A_{6,2}[\beta|\alpha]$, with $\alpha \neq \beta$. From these products, +66 are different from zero and are distributed as follows:

- There are +6 products, $D_5 \neq 0$, of the form

$$D_5 = -xy(x+y)^2(x^2 - xy + y^2)^2 y^2.$$

- There are +36 products, $D_4 \neq 0$, of the forms

$$D_4 = \begin{cases} -x^3y^5, & (18 \text{ cases}), \\ \text{or} & \\ x^4y^6, & (18 \text{ cases}). \end{cases}$$

- There are +18 products, $D_3 \neq 0$, of the form

$$D_3 = -x^3y^3.$$

- There are +6 products, $D_2 \neq 0$, of the form

$$D_2 = -xy^3.$$

Now we can state the following theorem.

Theorem 5.1. *Let the shift circulant matrix $A_{6,2}$ in (5.1). This matrix is sign symmetric if and only if $xy < 0$.*

Lemma 5.1 [6, Theorem 2.6]. *Let A be a sign symmetric $n \times n$ matrix. The following are equivalent:*

(i) *The matrix A is stable.*
(ii) *The matrix A is a P-matrix.*

Theorem 5.2. *Let $A_{6,2}$ be a sign symmetric shift circulant matrix. Then*

(i) $x > 0$

$x + y > 0 \iff A_{6,2}$ *is a P-matrix.*

(ii) $x < 0 \Rightarrow A_{6,2}$ *is not a P-matrix.*

Proof. The eigenvalues of $A_{6,2}$ are all double and the spectrum of $A_{6,2}$ is given by

$$\sigma(A_{6,2}) = \left\{ x+y, x - \frac{1}{2}y + \mathrm{i}\frac{\sqrt{3}}{2}y, x - \frac{1}{2}y - \mathrm{i}\frac{\sqrt{3}}{2}y \right\}. \tag{5.2}$$

Since $A_{6,2}$ is a sign symmetric shift circulant matrix, it is obvious from Theorem 5.1 that if $x > 0$ then $y < 0$ and so $x - \frac{1}{2}y > 0$. Therefore the matrix $A_{6,2}$ is a stable matrix if and only if $x + y > 0$. Lemma 5.1 proves the first part.

In case we have $x < 0$ it is $y > 0$ and then $x - \frac{1}{2}y < 0$. This proves that the matrix is neither a stable nor a P-matrix. □

Theorem 5.3. *Let $A_{6,2}$ be a shift circulant matrix, with $x, y \in IR$. This matrix is a P^2-matrix if and only if $x > 0, x + y > 0, x - y\sqrt[3]{2} > 0$.*

Proof. We denote by D_k the kth order principal determinant of the matrix $A_{6,2}$. Then a D_5 principal determinant results by eliminating a row and the corresponding column. From Fig. 3, in (I), we can see that all determinants are of the same type, since the graph of this is taken by removing one node. Then, with an appropriate permutation matrix P we have

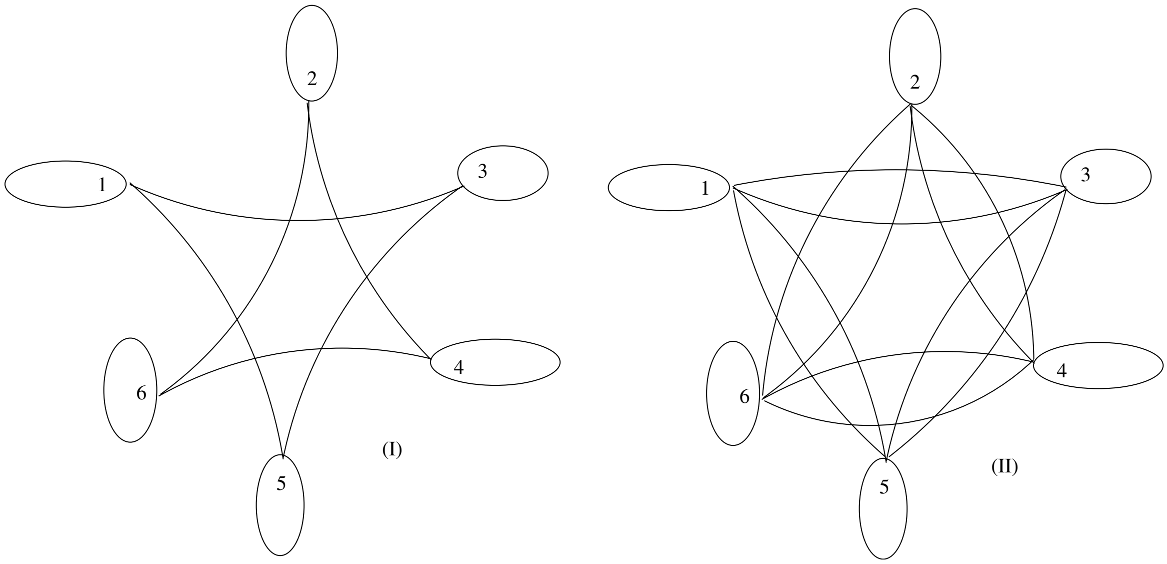

Fig. 3. The graph of the matrices $A_{6,2}$ and $A^2_{6,2}$.

$$D_5 = \begin{vmatrix} x & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ y & 0 & x & 0 & 0 \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & 0 & x \end{vmatrix} = x^2(x^3 + y^3). \tag{5.3}$$

Since two different graphs result from the removal of two nodes, there are two types of principal determinants D_4, which, by using an appropriate permutation matrix P, give

$$D_4 = \begin{vmatrix} x & y & 0 & 0 \\ 0 & x & y & 0 \\ y & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix} = x(x^3 + y^3) \quad \text{or} \quad D_4 = \begin{vmatrix} x & y & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \end{vmatrix} = x^4. \tag{5.4}$$

Since two different graphs result from the removal of three nodes, there are also two types of principal determinants D_3, which again, in a similar way, give

$$D_3 = \begin{vmatrix} x & y & 0 \\ 0 & x & y \\ y & 0 & x \end{vmatrix} = (x^3 + y^3) \quad \text{or} \quad D_3 = \begin{vmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = x^3. \tag{5.5}$$

Finally, we have

$$D_2 = x^2 \quad \text{and} \quad D_1 = x. \tag{5.6}$$

Let now

$$A^2_{6,2} = \begin{pmatrix} x^2 & 0 & 2xy & 0 & y^2 & 0 \\ 0 & x^2 & 0 & 2xy & 0 & y^2 \\ y^2 & 0 & x^2 & 0 & 2xy & 0 \\ 0 & y^2 & 0 & x^2 & 0 & 2xy \\ 2xy & 0 & y^2 & 0 & x^2 & 0 \\ 0 & 2xy & 0 & y^2 & 0 & x^2 \end{pmatrix}. \tag{5.7}$$

We denote by $\widehat{D}_k$ the kth order principal determinant of the matrix $A^2_{6,2}$. As before, from Fig. 3, in (II), we have that all determinants $\widehat{D}_5$ are the of same type and can be transformed by using an appropriate permutation matrix P to give

$$\widehat{D}_5 = \begin{vmatrix} x^2 & 2xy & y^2 & 0 & 0 \\ y^2 & x^2 & 2xy & 0 & 0 \\ 2xy & y^2 & x^2 & 0 & 0 \\ 0 & 0 & 0 & x^2 & 2xy \\ 0 & 0 & 0 & y^2 & x^2 \end{vmatrix} = x(x^3+y^3)^2(x^3-2y^3). \tag{5.8}$$

We determine the other principal determinants in an analogous way. So, we obtain

$$\widehat{D}_4 = \begin{cases} x^2(x^3+y^3)^2 \\ \text{or} \\ x^2(x^3-2y^3)^2 \end{cases}, \quad \widehat{D}_3 = \begin{cases} (x^3+y^3)^2 \\ \text{or} \\ x^3(x^3-2y^3) \end{cases}, \quad \widehat{D}_2 = \begin{cases} x^4 \\ \text{or} \\ x(x^3-2y^3) \end{cases}, \quad \widehat{D}_1 = x^2. \tag{5.9}$$

The above relationships prove the theorem. □

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