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More on modifications and improvements of classical iterative schemes for M-matrices

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Abstract

In the last four decades many articles have been devoted to the modifications and improvements of classes of preconditioners for linear systems whose matrix coefficient is an M-matrix in order to improve on the convergence rates of the classical iterative schemes (Jacobi, Gauss–Seidel, etc.). The present work is a contribution towards the generalization of the most common preconditioners used so far.

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1. Introduction and preliminaries

Consider the linear system of algebraic equations

$$Ax = b, \tag{1.1}$$

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where $A \in \mathbb{R}^{n,n}$ belongs to the class of Z-matrices (see [1]), that is its off-diagonal elements are nonpositive, $a_{ij} \leqslant 0,\ i, j = 1(1)n,$[1] $j \neq i$, and $b \in \mathbb{R}^n$. In this work we restrict to *nonsingular M-matrices* (see [1] and also [14,16]) that is to Z-matrices for which A^{-1} exists and $A^{-1} \geqslant 0$. Consider the usual *splitting* of A, namely

$$A = D - L - U, \tag{1.2}$$

where D is a diagonal matrix and L and U are strictly lower and strictly upper triangular matrices, respectively. Since A is a nonsingular M-matrix, $a_{ii} > 0,\ i = 1(1)n$, where to simplify the notation we may assume that $a_{ii} = 1,\ i = 1(1)n$. Under the assumptions so far, the iterative methods of Jacobi and of Gauss–Seidel associated with (1.1) converge and by virtue of the Stein–Rosenberg theorem [1,14,16] they exclusively hold

$$\text{either} \quad \rho(H) = \rho(B) = 0 \quad \text{or} \quad 0 < \rho(H) < \rho(B) < 1, \tag{1.3}$$

where $B := L + U,\ H := (I - L)^{-1}U$, the Jacobi and Gauss–Seidel iteration matrices, respectively, associated with $A = I - L - U$.

Many researchers have considered left preconditioners applied to system (1.1) that made the associated Jacobi and Gauss–Seidel (type) methods converge faster than the original ones. Such modifications or improvements based on prechosen preconditioners were considered by Milaszewicz [12] who based his idea on previous ones (see, e.g., [2,7,11]), by Gunawardena et al. [3], by Kohno et al. [8] who extended the main idea in [3], and very recently by Li and Sun [9] who extended the class of matrices considered in [8] and obtained many results for more general methods. The preconditioners considered in these works are presented below.

In a simpler form, Milaszewicz [12] considered the preconditioner

$$P_1 \equiv I + S_1 = \begin{bmatrix} 1 & & & & & & \\ -a_{21} & 1 & & & & & \\ -a_{31} & & 1 & & & & \\ \vdots & & & \ddots & & & \\ -a_{i1} & & & & 1 & & \\ \vdots & & & & & \ddots & \\ -a_{n1} & & & & & & 1 \end{bmatrix}, \tag{1.4}$$

which eliminates the elements of the first column of A below the diagonal. Gunawardena et al. [3] considered as a preconditioner the matrix

[1] The notation $a(b)c$ is an abbreviation of all the terms of the arithmetic progression with first term a, step $b > 0\ (< 0)$ and last term the largest (smallest) one that is not greater (smaller) than c.

$$P_2 \equiv I + S_2 = \begin{bmatrix} 1 & -a_{12} & & & & & \\ & 1 & -a_{23} & & & & \\ & & \ddots & \ddots & & & \\ & & & 1 & -a_{i,i+1} & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & -a_{n-1,n} \\ & & & & & & 1 \end{bmatrix}, \tag{1.5}$$

whose effect on A is to eliminate the elements of the first upper diagonal.

The selection of the above preconditioners P can be considered in a more general context as follows. To select a preconditioner for system (1.1) consider a matrix Q, with Q^{-1} approximating A^{-1}, by setting some off-diagonal elements of A to zero. When A is an M-matrix so will be Q, hence $Q^{-1} \geqslant 0$. Then, we can use as P, Q^{-1} itself or a part of it. In either case $P \geqslant 0$. So, if we wish to preserve the M-matrix character in PA, PA must be a Z-matrix. This can be done with P_1 or P_2, because, Q_1, in the case of P_1, is obtained from A by setting to zeros all off-diagonal but the elements of its first column, while $Q_2 = I - U$, $Q_2^{-1} = I + U + U^2 + \cdots + U^{n-1}$, and from the latter only its part that contains its diagonal and the first upper diagonal are retained in P_2.

Kohno et al. [8] parametrized P_2 above and considered as a preconditioner the matrix

$$P_2(\alpha) \equiv I + S_2(\alpha) = \begin{bmatrix} 1 & -\alpha_1 a_{12} & & & & & \\ & 1 & -\alpha_2 a_{23} & & & & \\ & & \ddots & \ddots & & & \\ & & & 1 & -\alpha_i a_{i,i+1} & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & -\alpha_{n-1} a_{n-1,n} \\ & & & & & & 1 \end{bmatrix}, \tag{1.6}$$

where $\alpha = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_{n-1}]^{\mathrm{T}} \in \mathbb{R}^{n-1}$, $\alpha_i \geqslant 0$, $i = 1(1)n-1$. Obviously, for $\alpha = 0 \in \mathbb{R}^{n-1}$, $P_2(\alpha) \equiv P_2(0) = I$, while for $\alpha = [1 \ 1 \ \cdots \ 1]^{\mathrm{T}} \in \mathbb{R}^{n-1}$, $P_2(\alpha) \equiv P_2(1) = P_2$. The same preconditioner was considered in [9] for $\alpha_i \in [0, 1]$, $i = 1(1)n-1$, with extensions of the splitting (1.2) to more general Gauss–Seidel type ones that gave many interesting results.

The outline of this work is as follows: In Section 2, we parametrize Milaszewicz's preconditioner by using the idea in [8] and base our Jacobi and Gauss–Seidel iterative schemes on the ones in [7,12] (see also [11]), sufficient conditions on the

α_i's are given that guarantee convergence and the "best", in some sense, set of the α_i's is found. In Section 3, similar work to that in Section 2 is done. Thus previous results, including some in [9], can be extended. In Section 4, numerical examples are presented in support of the theory developed. Finally, in Section 5, various points are made and some open questions are addressed.

2. Generalizing Milaszewicz's iterative schemes

The generalized (parametrized) preconditioner used in this case is of the form

$$P_1(\alpha) \equiv I + S_1(\alpha) = \begin{bmatrix} 1 & & & & & & \\ -\alpha_2 a_{21} & 1 & & & & & \\ -\alpha_3 a_{31} & & 1 & & & & \\ \vdots & & & \ddots & & & \\ -\alpha_i a_{i1} & & & & 1 & & \\ \vdots & & & & & \ddots & \\ -\alpha_n a_{n1} & & & & & & 1 \end{bmatrix}, \tag{2.1}$$

where $\alpha = [\alpha_2 \;\; \alpha_3 \;\; \cdots \;\; \alpha_n]^{\mathrm{T}} \in \mathbb{R}^{n-1}$ with α_i, $i = 2(1)n$, constants. The selection of α's will be made from the $(n-1)$-dimensional nonnegative cone K_{n-1} in such a way that none of the diagonal elements of the preconditioned matrix $\tilde{A}(\alpha) := P_1(\alpha)A$ vanishes.

The following sets of integers will be useful in our analysis.

$$N := \{1, 2, \ldots, n\}, \quad N_1 := N\backslash\{1\}, \quad N_2 := \{i \in N_1 : a_{i1} \neq 0\}. \tag{2.2}$$

Additional assumption: In the matrices A we consider we assume that there is at least a pair of indices $i, j \in N_1$, such that $a_{i1}a_{1j} \neq 0$, so $\tilde{A}(\alpha)$ differs from A.

2.1. Convergence of the Jacobi and Gauss–Seidel iterative schemes

Applying $P_1(\alpha)$ on (1.1) we obtain the equivalent linear system

$$\tilde{A}(\alpha)x = \tilde{b}(\alpha), \quad \text{with } \tilde{A}(\alpha) = \big(I + S_1(\alpha)\big)A, \quad \tilde{b}(\alpha) = \big(I + S_1(\alpha)\big)b, \tag{2.3}$$

where, if needed, we will write

$$\tilde{A}(\alpha) = \widetilde{D}(\alpha) - \tilde{L}(\alpha) - \tilde{U}(\alpha), \tag{2.4}$$

with $\widetilde{D}(\alpha)$ diagonal and $\tilde{L}(\alpha)$ and $\tilde{U}(\alpha)$ strictly lower and strictly upper triangular matrices. Note that in view of (1.2) and (2.3)

$$\tilde{A}(\alpha) = I - L - U + S_1(\alpha) - S_1(\alpha)L - S_1(\alpha)U,$$

with $S_1(\alpha)L = 0$. The elements $\tilde{a}_{ij}(\alpha)$ of $\tilde{A}(\alpha)$ are given by the expressions:

$$\tilde{a}_{ij}(\alpha) = \begin{cases} a_{1j}, & i = 1, \quad j \in N, \\ (1-\alpha_i)a_{i1}, & i \in N_1, \quad j = 1, \\ a_{ij} - \alpha_i a_{i1} a_{1j}, & i, j \in N_1. \end{cases} \tag{2.5}$$

Requesting that $\tilde{a}_{i1}(\alpha) = (1-\alpha_i)a_{i1} \leqslant 0$, $i \in N_1$, the nonpositivity of all the off-diagonal elements will be preserved and so will be the Z-matrix character of $\tilde{A}(\alpha)$. So, if $i \in N_2$ then $\alpha_i \leqslant 1$. If $i \in N_1 \backslash N_2$ then any value for α_i will do since the ith row of A will remain unchanged. To guarantee positivity for $\tilde{a}_{ii}(\alpha)$, $1 - \alpha_i a_{i1} a_{1i} > 0$, a condition covered by the previous one ($\alpha_i \leqslant 1$), since for a Z-matrix the statement "A is a nonsingular M-matrix" is equivalent to the statement "all the principal minors of A are positive" (see Theorem 6.2.3, Condition (A_1) of [1]) implying $a_{i1}a_{1i} < 1$, $i \in N_1$.

In view of the discussion just made we restrict to $\alpha_i \in [0, 1]$, $i \in N_2$.

Case I. $\alpha_i \in [0, 1]$, $i \in N_2$. Defining the matrices

$$\begin{aligned} &D_\alpha := \operatorname{diag}\big(0,\ \alpha_2 a_{21} a_{12},\ \ldots,\ \alpha_n a_{n1} a_{1n}\big) \quad \text{and} \\ &S_1(\alpha)U = (P_1(\alpha) - I)\, U := L_\alpha + D_\alpha + U_\alpha, \end{aligned} \tag{2.6}$$

where L_α, U_α the strictly lower and strictly upper triangular components of $S_1(\alpha)U$, then, from (2.5), the restrictions on a_{ij}'s and α_i's, the fact that $S_1(\alpha)L = 0$, and the preceding discussion, the three matrices on the right hand side of (2.4) are given by

$$\widetilde{D}(\alpha) = I - D_\alpha, \quad \tilde{L}(\alpha) = L - S_1(\alpha) + L_\alpha, \quad \tilde{U}(\alpha) = U + U_\alpha. \tag{2.7}$$

The diagonal elements of $\widetilde{D}(\alpha)$ are positive while those of $\tilde{L}(\alpha)$ and $\tilde{U}(\alpha)$ are nonnegative.

For the needs of one of our main statements the following splittings will be considered:

$$\tilde{A}(\alpha) = \begin{cases} M(\alpha) - N(\alpha) = (I + S_1(\alpha)) - (I + S_1(\alpha))(L + U), \\ M'(\alpha) - N'(\alpha) = I - (L - S_1(\alpha) + L_\alpha + D_\alpha + U + U_\alpha), \\ M''(\alpha) - N''(\alpha) = (I - D_\alpha) - (L - S_1(\alpha) + L_\alpha + U + U_\alpha). \end{cases} \tag{2.8}$$

Below we define the Jacobi type iteration matrices associated with the above splittings:

$$\begin{aligned} B(\alpha) \equiv B &:= M^{-1}(\alpha)N(\alpha) = L + U, \\ B'(\alpha) &:= {M'}^{-1}(\alpha)N'(\alpha) = I - (I + S_1(\alpha))A \\ &= L - S_1(\alpha) + D_\alpha + L_\alpha + U + U_\alpha, \end{aligned}$$

$$\begin{aligned}\tilde{B}(\alpha) \equiv B''(\alpha) &:= M''^{-1}(\alpha)N''(\alpha)\\ &= (I - D_\alpha)^{-1}(I - (I + S_1(\alpha))A - D_\alpha)\\ &= (I - D_\alpha)^{-1}(L - S_1(\alpha) + L_\alpha + U + U_\alpha),\end{aligned} \tag{2.9}$$

as well as the splittings that define the Gauss–Seidel type matrices:

$$\tilde{A}(\alpha) = \begin{cases} M(\alpha) - N(\alpha) = (I - (L - S_1(\alpha))) - (I + S_1(\alpha))U, \\ M'(\alpha) - N'(\alpha) = ((I - (L - S_1(\alpha))) - L_\alpha) - (D_\alpha + U + U_\alpha), \\ M''(\alpha) - N''(\alpha) = ((I - (L - S_1(\alpha))) - L_\alpha - D_\alpha) - (U + U_\alpha), \end{cases} \tag{2.10}$$

$$\begin{aligned} &H(\alpha) \equiv H := (I - L)^{-1}U,\\ &H'(\alpha) := ((I - (L - S_1(\alpha))) - L_\alpha)^{-1}(D_\alpha + U + U_\alpha),\\ &\widetilde{H}(\alpha) \equiv H''(\alpha) := ((I - (L - S_1(\alpha))) - D_\alpha - L_\alpha)^{-1}(U + U_\alpha).\end{aligned} \tag{2.11}$$

Theorem 2.1. (a) *Under the assumptions and the notation so far, for any $\alpha \in K_{n-1}$ such that $\alpha_i \in [0, 1]$, $i \in N_2$, there hold*:

There exists $y \in \mathbb{R}^n$, with $y \geqslant 0$, such that

$$B'(\alpha)y \leqslant By, \tag{2.12}$$

$$\rho\big(\tilde{B}(\alpha)\big) \leqslant \rho\big(B'(\alpha)\big) < 1, \tag{2.13}$$

$$\rho\big(\widetilde{H}(\alpha)\big) \leqslant \rho\big(H'(\alpha)\big) \leqslant \rho(H) < 1, \tag{2.14}$$

$$\rho\big(\widetilde{H}(\alpha)\big) \leqslant \rho\big(\tilde{B}(\alpha)\big), \quad \rho(H'(\alpha)) \leqslant \rho(B'(\alpha)), \quad \rho(H) < \rho(B) < 1. \tag{2.15}$$

(*Notes*: (i) *Equalities in* (2.15) *hold if and only if* $\rho(B) = 0$. (ii) *In* [13] *it is proved that* (2.12) *implies* $\rho(B'(\alpha)) \leqslant \rho(B)$.)

(b) *Suppose that A is irreducible. Then*:

(i) *For $\alpha_i \in [0, 1)$, $i \in N_2$, provided that $\alpha \neq 0$, the matrices $\tilde{B}(\alpha)$, $B'(\alpha)$ and B are irreducible and all the inequalities in* (2.13)–(2.15) *are strict. Moreover, there holds*

$$\rho(B'(\alpha)) \leqslant \rho(B). \tag{2.16}$$

(ii) *For $\alpha_i = 1$, $i \in N_2$, the $(n - 1) \times (n - 1)$ matrices $B_1'(1)$ and $\tilde{B}_1(1)$ of the bottom right corner of $B'(1)$ and $\tilde{B}(1)$ are irreducible and all the inequalities in* (2.13)–(2.16) *are strict.*

Proof. (a) (2.12): To prove (2.12) we need the expressions of the nonnegative elements of the two Jacobi iteration matrices involved. Below we give the elements for all three matrices in (2.9):

$$b_{ii} = 0, \ \ i \in N, \quad b_{ij} = -a_{ij}, \ \ i, j \in N, \ \ j \neq i, \tag{2.17}$$

$$\begin{cases} b'_{ii}(\alpha) = \alpha_i a_{i1} a_{1i} = \alpha_i b_{i1} b_{1i}, & i \in N_2, \\ b'_{ii}(\alpha) = 0, & i \in N \backslash N_2, \\ b'_{ij}(\alpha) = -a_{ij} = b_{ij}, & i \in N \backslash N_2, \ \ j \in N_1, \ \ j \neq i, \\ b'_{i1}(\alpha) = (\alpha_i - 1) a_{i1} = (1 - \alpha_i) b_{i1}, & i \in N_2, \\ b'_{ij}(\alpha) = \alpha_i a_{i1} a_{1j} - a_{ij} = \alpha_i b_{i1} b_{1j} + b_{ij}, & i \in N_2, \ \ j \in N_1, \ \ j \neq i, \end{cases} \tag{2.18}$$

and

$$\begin{cases} \tilde{b}_{ii}(\alpha) = 0, & i \in N, \\ \tilde{b}_{ij}(\alpha) = -a_{ij} = b_{ij}, & i \in N \backslash N_2, \ \ j \in N_1, \ \ j \neq i, \\ \tilde{b}_{i1}(\alpha) = \dfrac{(\alpha_i - 1) a_{i1}}{1 - \alpha_i a_{i1} a_{1i}} = \dfrac{(1 - \alpha_i) b_{i1}}{1 - \alpha_i b_{i1} b_{1i}}, & i \in N_2, \\ \tilde{b}_{ij}(\alpha) = \dfrac{\alpha_i a_{i1} a_{1j} - a_{ij}}{1 - \alpha_i a_{i1} a_{1i}} = \dfrac{\alpha_i b_{i1} b_{1j} + b_{ij}}{1 - \alpha_i b_{i1} b_{1i}}, & i \in N_2, \ \ j \in N_1, \ \ j \neq i. \end{cases} \tag{2.19}$$

For the nonnegative Jacobi iteration matrix B there exists a nonnegative vector y such that $By = \rho(B)y$. Equating the ith rows, for $i \in N_2$, of the two vectors and replacing the elements b_{ij} of B in terms of the elements $b'_{ij}(\alpha)$ of $B'(\alpha)$ using (2.17) and (2.18) we successively obtain

$$\begin{aligned} \rho(B) y_i &= \sum_{j=1, j \neq i}^{n} b_{ij} y_j = b_{i1} y_1 + \sum_{j=2, j \neq i}^{n} b_{ij} y_j \\ &= \big(b'_{i1}(\alpha) + \alpha_i b_{i1}\big) y_1 + \sum_{j=2, j \neq i}^{n} \big(b'_{ij}(\alpha) - \alpha_i b_{i1} b_{1j}\big) y_j \\ &\quad + b'_{ii}(\alpha) y_i - b'_{ii}(\alpha) y_i \\ &= \sum_{j=1}^{n} b'_{ij}(\alpha) y_j - \alpha_i b_{i1} \sum_{j=2, j \neq i}^{n} b_{1j} y_j \\ &\quad - \alpha_i b_{i1} b_{1i} y_i + \alpha_i b_{i1} y_1 \\ &= \sum_{j=1}^{n} b'_{ij}(\alpha) y_j - \alpha_i b_{i1} \sum_{j=2}^{n} b_{1j} y_j + \alpha_i b_{i1} y_1. \end{aligned} \tag{2.20}$$

Using the fact that $\rho(B)y_1 = \sum_{j=2}^{n} b_{1j}y_j$ and replacing in (2.20) we have that

$$\rho(B)y_i = \sum_{j=1}^{n} b'_{ij}(\alpha)y_j + \alpha_i b_{i1}\left(\frac{1}{\rho(B)} - 1\right)\sum_{j=2}^{n} b_{1j}y_j. \tag{2.21}$$

Since the second term on the sum in (2.21) is nonnegative we have that

$$\sum_{j=1}^{n} b'_{ij}(\alpha)y_j \leqslant \sum_{j=1}^{n} b_{ij}y_j \tag{2.22}$$

from which (2.12) follows.

(a) (2.13): For a Z-matrix A the statement "A is a nonsingular M-matrix" is equivalent to the statement "there exists a positive vector $y\ (>0) \in \mathbb{R}^n$, that is $y_i > 0$, $i \in N$, such that $Ay > 0$" (see Theorem 6.2.3, Condition (I_{27}) of [1]). But $P_1(\alpha) = I + S_1(\alpha) \geqslant 0$, implies $\tilde{A}(\alpha)y = P_1(\alpha)Ay > 0$. Consequently, $\tilde{A}(\alpha)$, which is a Z-matrix, is a nonsingular M-matrix. So, the last two splittings in (2.8) are regular ones because $M'^{-1}(\alpha) = I^{-1} = I \geqslant 0$, $N'(\alpha) \geqslant 0$ and $M''^{-1}(\alpha) = (I - D_\alpha)^{-1} \geqslant 0$, $N''(\alpha) \geqslant 0$ and so they are convergent. Since $M''^{-1}(\alpha) \geqslant M'^{-1}(\alpha)$, it is implied [15] that the left inequality in (2.13) is true.

(a) (2.14): Consider the splittings (2.10) that define the iteration matrices in (2.11). The matrix $M(\alpha) = I - (L - S_1(\alpha))$ of the first splitting is lower triangular with units on the diagonal, elements of the first column $(1 - \alpha_i)a_{i1}$, $i \in N_1$, and remaining ones those of the strictly lower triangular part of A (a_{ij}, $i \in N_1\backslash\{2\}$, $j \in N_1$, $j < i$). So, all the off-diagonal elements of $M(\alpha)$ are nonpositive and therefore $M(\alpha)$ is a nonsingular M-matrix which implies that $M^{-1}(\alpha) \geqslant 0$. Also, $(I + S_1(\alpha))U \geqslant 0$, so the first splitting in (2.10) is a regular one. $M'(\alpha)$ can be written as $M'(\alpha) = M(\alpha) - L_\alpha = M(\alpha)(I - M^{-1}(\alpha)L_\alpha)$, and setting $\overline{L} = M^{-1}(\alpha)L_\alpha \geqslant 0$ we have

$$M'^{-1}(\alpha) = \left(I - \overline{L}\right)^{-1}M^{-1}(\alpha) = \left(I + \overline{L} + \overline{L}^2 + \cdots + \overline{L}^{n-1}\right)M^{-1}(\alpha) \geqslant 0. \tag{2.23}$$

Since $N'(\alpha) = D_\alpha + U + U_\alpha \geqslant 0$, the second splitting in (2.10) is also a regular one. The last splitting is a regular one since $\tilde{A}(\alpha)$ is a nonsingular M-matrix and so is $M''(\alpha)$ since the latter is derived from the former by setting some off-diagonal elements equal to zero (Theorem 3.12 of [14]) and $N''(\alpha) = U + U_\alpha \geqslant 0$. The inequalities in (2.14) are established because we notice that $N(\alpha) = L_\alpha + D_\alpha + U + U_\alpha \geqslant N'(\alpha) = D_\alpha + U + U_\alpha \geqslant N''(\alpha) = U + U_\alpha$.

(a) (2.15): Since A is a nonsingular M-matrix, the rightmost inequality is a straightforward implication of the Stein–Rosenberg Theorem as was mentioned before. The other two inequalities in (2.15) are implied directly by the facts that $\tilde{A}(\alpha)$ is a nonsingular M-matrix, and the last two pairs of splittings in (2.8) and

(2.10), from which the four matrices involved, $\widetilde{H}(\alpha)$, $\tilde{B}(\alpha)$, $H'(\alpha)$, $B'(\alpha)$, are produced, are regular ones with $L - S_1(\alpha) + L_\alpha + U + U_\alpha \geqslant U + U_\alpha$ and $L - S_1(\alpha) + L_\alpha + D_\alpha + U + U_\alpha \geqslant D_\alpha + U + U_\alpha$.

(b) For $\alpha_i \in [0, 1)$, $\tilde{A}(\alpha)$ is irreducible because it inherits the nonzero structure of the irreducible matrix A.

(bi) (2.13)–(2.16): By virtue of the irreduciblity of the corresponding matrices involved, the theorems used previously are also applied to prove the strict inequalities in (2.13)–(2.15), while (2.16) is proved in Theorem 2.2 of [12].

(bii) We consider the block partitionings

$$A = \left[\begin{array}{c|c} 1 & a_h^{\mathrm{T}} \\ \hline a_v & A_1 \end{array}\right], \quad P_1(1) = \left[\begin{array}{c|c} 1 & 0_{n-1}^{\mathrm{T}} \\ \hline -a_v & I_1 \end{array}\right],$$
$$\tilde{A}(1) = \left[\begin{array}{c|c} 1 & a_h^{\mathrm{T}} \\ \hline 0_{n-1} & \tilde{A}_1(1) \end{array}\right]. \tag{2.24}$$

Then the associated block Jacobi and Gauss–Seidel iteration matrices will be

$$B = \left[\begin{array}{c|c} 0 & -a_h^{\mathrm{T}} \\ \hline -a_v & B_1 \end{array}\right], \quad B'(1) = \left[\begin{array}{c|c} 0 & -a_h^{\mathrm{T}} \\ \hline 0_{n-1} & B_1'(1) \end{array}\right],$$
$$\tilde{B}(1) = \left[\begin{array}{c|c} 0 & -a_h^{\mathrm{T}} \\ \hline 0_{n-1} & \tilde{B}_1(1) \end{array}\right], \tag{2.25}$$

and

$$H = \left[\begin{array}{c|c} 0 & -a_h^{\mathrm{T}} \\ \hline 0_{n-1} & H_1 \end{array}\right], \quad H'(1) = \left[\begin{array}{c|c} 0 & -a_h^{\mathrm{T}} \\ \hline 0_{n-1} & H_1'(1) \end{array}\right],$$
$$\widetilde{H}(1) = \left[\begin{array}{c|c} 0 & -a_h^{\mathrm{T}} \\ \hline 0_{n-1} & \widetilde{H}_1(1) \end{array}\right]. \tag{2.26}$$

(bii) (2.13)–(2.16): By studying the structure of the matrices B_1, $B_1'(1)$, $\tilde{B}_1(1)$, H_1, $H_1'(1)$ and $\widetilde{H}_1(1)$ we can find out that the associated irreducibility properties will hold for these matrices. So, the theorems used previously are also applied in each case to prove the strict inequalities (2.13)–(2.15) while (2.16) is proved in Theorem 2.2 of [12]. $\square$

Below we give a lemma which is essentially proved in [13], extends Lemma 3.3 of [10] and gives a third alternative to Theorems 3.13 and 3.15 of [10].

Lemma 2.1. *Let $A_1, A_2 \in \mathbb{R}^{n,n}$ and that $A_i = M_i - N_i,\ i = 1, 2$, are weak splittings ($T_i = M_i^{-1}N_i \geqslant 0,\ i = 1, 2$). If the Perron eigenvector z_2 ($\geqslant 0$) of T_2 satisfies $T_1z_2 \leqslant T_2z_2$ then $\rho(T_1) \leqslant \rho(T_2)$.*

Proof. If T_2 is irreducible, $z_2 > 0$. Since $T_2z_2 = \rho(T_2)z_2$ it is $T_1z_2 \leqslant \rho(T_2)z_2$ and Lemma 3.3 of [10] applies. Hence $\rho(T_1) \leqslant \rho(T_2)$. If T_2 is reducible, $z_2 \geqslant 0$. Replace some zeros in T_2 by $\epsilon > 0$ so that $T_2(\epsilon)$ is irreducible. It will then be $T_1z_2(\epsilon) \leqslant \rho(T_2(\epsilon))z_2(\epsilon)$. Because of the continuous dependence of $\rho(T_2(\epsilon))$ and $z_2(\epsilon)$ on ϵ, taking limits as $\epsilon \to 0^+$ the same conclusion as before derives. □

We can prove that the spectral radii of the Jacobi and the Gauss–Seidel iteration matrices, $\tilde{B}(\alpha)$ and $\widetilde{H}(\alpha)$, are nonincreasing functions of any $\alpha_i \in [0, 1],\ i \in N_2$. Specifically:

Theorem 2.2. *Under the assumptions and the notation so far let*

$$[0\ \ 0\ \ \cdots\ \ 0]^{\mathrm{T}} \leqslant \alpha \leqslant \alpha' \leqslant [1\ \ 1\ \ \cdots\ \ 1]^{\mathrm{T}} \in K_{n-1}. \tag{2.27}$$

Then

$$\rho\big(\tilde{B}(\alpha')\big) \leqslant \rho\big(\tilde{B}(\alpha)\big) \quad \textit{and} \quad \rho\big(\widetilde{H}(\alpha')\big) \leqslant \rho\big(\widetilde{H}(\alpha)\big). \tag{2.28}$$

Proof. Note that the Jacobi and the Gauss–Seidel iteration matrices associated with any $A = D - L - U$ (D invertible diagonal, L and U strictly lower and upper triangular), are the same with those associated with $D^{-1}A = I - D^{-1}L - D^{-1}U$. Next, observe that by virtue of Lemma 2.1, the nature of the vector y in (2.12), and (2.13), it is $\rho(\tilde{B}(\alpha)) \leqslant \rho(B)$. From (2.14), $\rho(\widetilde{H}(\alpha)) \leqslant \rho(H)$. Therefore, the Jacobi and the Gauss–Seidel iterative methods associated with a preconditioned matrix $\tilde{A}(\alpha)$, with α as in (2.27), are no worse than the corresponding ones of the unpreconditioned matrix A. Since $\widetilde{D}^{-1}\tilde{A}$ has the same Jacobi and Gauss–Seidel iteration matrices with $\tilde{A}$, its elements, denoted by the same symbols as those of $\tilde{A}$, are:

$$\begin{aligned}
&\tilde{a}_{ii} = 1,\ \ i \in N, \quad \tilde{a}_{1j} = a_{1j},\ \ j \in N_1,\\
&\tilde{a}_{i1} = \frac{(1-\alpha_i)a_{i1}}{1-\alpha_i a_{i1}a_{1i}},\ \ i \in N_1,\\
&\tilde{a}_{ij} = \frac{a_{ij} - \alpha_i a_{i1}a_{1j}}{1-\alpha_i a_{i1}a_{1i}},\ \ i \in N_1,\ \ j \in N_1\backslash\{i\}.
\end{aligned} \tag{2.29}$$

Consider the vector $\beta \in K_{n-1}$ whose components are defined by

$$\beta_i = 0,\ \ \text{if}\ \alpha_i = 1 \quad \text{and} \quad \beta_i = \frac{\alpha_i' - \alpha_i}{1-\alpha_i}\ (\in [0, 1]),\ \ \text{if}\ \alpha_i \neq 1.$$

Apply to $\widetilde{D}^{-1}\tilde{A}$ the preconditioner $P_1(\beta)$. The Jacobi and the Gauss–Seidel iterative methods associated with the new preconditioned matrix $\tilde{\tilde{A}}(\beta) = P_1(\beta)\widetilde{D}^{-1}\tilde{A}$ will be no worse than the ones corresponding to $\widetilde{D}^{-1}\tilde{A}$. The elements $\tilde{\tilde{a}}_{ij}$ of the matrix $\tilde{\tilde{D}}^{-1}(\beta)\tilde{\tilde{A}}(\beta)$ will be given by the same expressions as those in (2.29) where the α_{ij}'s will be replaced by $\tilde{\alpha}_{ij}$'s and the α_i's by β_i's. The $\tilde{\tilde{a}}_{ij}$'s are given by

$$\begin{aligned}
&\tilde{\tilde{a}}_{ii} = 1,\ \ i \in N, \quad \tilde{\tilde{a}}_{1j} = \tilde{a}_{1j}, \ \ j \in N_1,\\
&\tilde{\tilde{a}}_{i1} = \frac{(1-\beta_i)\tilde{a}_{i1}}{1-\beta_i\tilde{a}_{i1}\tilde{a}_{1i}}, \ \ i \in N_1,\\
&\tilde{\tilde{a}}_{ij} = \frac{\tilde{a}_{ij}-\beta_i\tilde{a}_{i1}\tilde{a}_{1j}}{1-\beta_i\tilde{a}_{i1}\tilde{a}_{1i}}, \ \ i \in N_1, \ \ j \in N_1\backslash\{i\}.
\end{aligned} \tag{2.30}$$

Substituting in (2.30) the $\tilde{a}_{ij}$'s and the β_i's, after some simple algebra, we end up with:

$$\begin{aligned}
&\tilde{\tilde{a}}_{ii} = 1,\ \ i \in N, \quad \tilde{\tilde{a}}_{1j} = a_{1j}, \ \ j \in N_1,\\
&\tilde{\tilde{a}}_{i1} = \frac{(1-\alpha_i')a_{i1}}{1-\alpha_i' a_{i1}a_{1i}}, \ \ i \in N_1,\\
&\tilde{\tilde{a}}_{ij} = \frac{a_{ij}-\alpha_i' a_{i1}a_{1j}}{1-\alpha_i' a_{i1}a_{1i}}, \ \ i \in N_1, \ \ j \in N_1\backslash\{i\},
\end{aligned} \tag{2.31}$$

which effectively proves (2.28). □

If any $\alpha_i > 1,\ i \in N_2$, then elements of the Jacobi iteration matrix $\tilde{B}(\alpha)$ are negative and the theory of Case I can not be applied. This forces us to restrict the class of the invertible M-matrices A to the strictly diagonally dominant (SDD) and the irreducibly diagonally dominant (IDD) matrices. For this, some further notation and terminology are introduced.

For the splitting (1.2), the *comparison matrix*, $\mathscr{M}(A)$, of $A \in \mathbb{C}^{n,n}$ is the matrix

$$\mathscr{M}(A) = |D| - |L| - |U|, \tag{2.32}$$

where $|\cdot|$ denotes the matrix whose elements are the moduli of the elements of the given matrix. $A \in \mathbb{C}^{n,n}$ is an *H-matrix* if and only if its comparison matrix is an M-matrix. Also, for reasons which will become clear soon, we define the quantities below

$$\begin{aligned}
&d_i = |a_{ii}|,\ \ i \in N, \quad l_1 = 0, \quad l_i = \sum_{j=1}^{i-1} |a_{ij}|,\ \ i \in N_1,\\
&u_i = \sum_{j=i+1}^{n} |a_{ij}|,\ \ i \in N\backslash\{n\}, \quad u_n = 0,
\end{aligned} \tag{2.33}$$

and assume that

$$|l_i| + |u_i| > 0,\ i \in N, \tag{2.34}$$

so that in each row there is at least one off-diagonal element different from zero; for otherwise we would practically have a linear system of $n-1$ equations with $n-1$ unknowns to solve.

As in [8], we set

$$p_i = a_{i1}a_{1i}, \quad q_i = a_{i1}\sum_{j=1}^{i-1} a_{1j}, \quad r_i = a_{i1}\sum_{j=i+1}^{n} a_{1j}, \tag{2.35}$$

so

$$p_i + q_i + r_i = a_{i1}\sum_{j=1}^{n} a_{1j} = a_{i1}(1 - l_1 - u_1) \leqslant 0, \ \ i \in N_2. \tag{2.36}$$

For the Jacobi and the Gauss–Seidel iteration matrices, $\tilde{B}(\alpha)$ and $\widetilde{H}(\alpha)$, to converge sufficient conditions are (see, e.g., [5,6])

$$\begin{aligned} \rho\big(\tilde{B}(\alpha)\big) &\leqslant \max_{i\in N} \frac{\tilde{l}_i(\alpha) + \tilde{u}_i(\alpha)}{\tilde{d}_i(\alpha)} < 1, \\ \rho\big(\widetilde{H}(\alpha)\big) &\leqslant \max_{i\in N} \frac{\tilde{u}_i(\alpha)}{\tilde{d}_i(\alpha) - \tilde{l}_i(\alpha)} < 1, \end{aligned} \tag{2.37}$$

which will be used whenever needed.

Case II. $\alpha_i > 1, \ i \in N_2$. In the sequel we examine the case where all α_i's take values greater than 1 without destroying the positivity of the diagonal elements $\tilde{a}_{ii}(\alpha)$ and preserving at the same time for each row of the comparison matrix $\mathscr{M}(\tilde{A}(\alpha))$ of $\tilde{A}(\alpha)$, at least, the inequalities that the corresponding row of A satisfies.

For $\tilde{a}_{ii}(\alpha) > 0, \ i \in N_2$, to hold there must be $\alpha_i a_{i1}a_{1i} < 1$ because for $i \in N\backslash N_2, \tilde{a}_{ii}(\alpha) = a_{ii} = 1 > 0$. Since A is an M-matrix, and therefore $a_{i1}a_{1i} < 1$, $\tilde{a}_{ii}(\alpha) > 0$ implies

$$\alpha_i \in \begin{cases} \left(1, \dfrac{1}{a_{i1}a_{1i}}\right) & \text{if } a_{i1}a_{1i} \neq 0, \\ (1, \infty) & \text{if } a_{i1}a_{1i} = 0, \end{cases} \quad i \in N_2. \tag{2.38}$$

On the other hand, if (2.38) hold then

$$\tilde{a}_{ij}(\alpha) \begin{cases} > 0 & \text{if } i = j \in N, \\ \geqslant 0 & \text{if } i \in N_1, \ \ j = 1, \\ \leqslant 0 & \text{if } i \in N_1, \ \ j \in N\backslash\{1, i\}. \end{cases} \tag{2.39}$$

In the present case it is

$$\begin{aligned}
\tilde{d}_i(\alpha) &= |\tilde{a}_{ii}(\alpha)| = \tilde{a}_{ii}(\alpha) = 1 - \alpha_i a_{i1} a_{1i} = 1 - \alpha_i p_i,\\
\tilde{l}_i(\alpha) &= \sum_{j=1}^{i-1} |\tilde{a}_{ij}(\alpha)| = \tilde{a}_{i1}(\alpha) - \sum_{j=2}^{i-1} \tilde{a}_{ij}(\alpha)\\
&= a_{i1} - \alpha_i a_{i1} + \alpha_i a_{i1} \sum_{j=2}^{i-1} a_{1j} - \sum_{j=2}^{i-1} a_{ij}\\
&= 2a_{i1} + l_i - 2\alpha_i a_{i1} + \alpha_i q_i,\\
\tilde{u}_i(\alpha) &= \sum_{j=i+1}^{n} |\tilde{a}_{ij}(\alpha)| = - \sum_{j=i+1}^{n} \tilde{a}_{ij}(\alpha)\\
&= \alpha_i a_{i1} \sum_{j=i+1}^{n} a_{1j} - \sum_{j=i+1}^{n} a_{ij} = \alpha_i r_i + u_i,
\end{aligned} \tag{2.40}$$

hence, by (2.40), (2.35) and (2.36),

$$\tilde{d}_i(\alpha) - \tilde{l}_i(\alpha) - \tilde{u}_i(\alpha) \;=\; 1 - l_i - u_i - 2a_{i1} + \alpha_i a_{i1}(1 + l_1 + u_1). \tag{2.41}$$

Requiring to always have $\tilde{d}_i(\alpha) > \tilde{l}_i(\alpha) + \tilde{u}_i(\alpha)$ we obtain $\alpha_i(-a_{i1})(1 + l_1 + u_1) < 1 - l_i - u_i - 2a_{i1}$. For $i \in N_2$, since $1 + l_1 + u_1 > 0$, there must be

$$\alpha_i < \frac{1 - l_i - u_i - 2a_{i1}}{(-a_{i1})(1 + l_1 + u_1)}, \tag{2.42}$$

provided that the expression in (2.42) is greater than 1. However,

$$\frac{1 - l_i - u_i - 2a_{i1}}{(-a_{i1})(1 + l_1 + u_1)} \geqslant \frac{-2a_{i1}}{(-a_{i1})(1 + l_1 + u_1)} = \frac{2}{1 + l_1 + u_1} > 1. \tag{2.43}$$

If $a_{1i} = 0$, the upper bound for the value of α_i is not ∞ as (2.38) indicates but the expression given in (2.42). If $a_{1i} \neq 0$, the upper bound for α_i should be the smallest of $1/(a_{i1}a_{1i})$ in (2.38) and the one in (2.42) above. But

$$\begin{aligned}
&\frac{1}{a_{i1}a_{1i}} - \frac{1 - l_i - u_i - 2a_{i1}}{(-a_{i1})(1 + l_1 + u_1)}\\
&\quad = \frac{(1 - a_{i1}a_{1i}) + (l_1 + u_1 + a_{1i}) + (-a_{1i})(l_i + u_i + a_{i1})}{a_{i1}a_{1i}(1 + l_1 + u_1)} > 0,
\end{aligned}$$

since all the terms in the numerator are nonnegative with the first one positive. So, from (2.42)

$$\alpha_i \in I_i := \left(1, \frac{1 - l_i - u_i - 2a_{i1}}{(-a_{i1})(1 + l_1 + u_1)}\right), \quad i \in N_2. \tag{2.44}$$

After the analysis just given it is implied that the comparison matrix $\mathscr{M}(\tilde{A}(\alpha))$ of $\tilde{A}(\alpha)$ for all $\alpha_i \in I_i$, $i \in N_2$, is a Z-matrix with the same nonzero pattern as that of A. Also, it has row sums positive even in cases where A had corresponding sums zero. Consequently, $\mathscr{M}(\tilde{A}(\alpha))$ is a nonsingular M-matrix. Also, if A is irreducible, so is $\mathscr{M}(\tilde{A}(\alpha))$.

In view of (2.39) the off-diagonal elements of $\tilde{A}(\alpha)$ are *not* all nonpositive. This suggests that a direct conclusion regarding convergence of the iteration matrices considered in Case I can *not* be drawn. So, we turn our attention to the comparison matrix $\mathscr{M}(\tilde{A}(\alpha))$ of $\tilde{A}(\alpha)$ and its associated Jacobi, Jacobi type, Gauss–Seidel and Gauss–Seidel type iteration matrices. First, we consider the main splitting of $\mathscr{M}(\tilde{A}(\alpha))$, as in (2.4) for $\tilde{A}(\alpha)$, which, because $\widetilde{D}(\alpha)$, $\tilde{U}(\alpha) \geqslant 0$, is

$$\mathscr{M}(\tilde{A}(\alpha)) = \widetilde{D}(\alpha) - |\tilde{L}(\alpha)| - \tilde{U}(\alpha). \tag{2.45}$$

We consider the Jacobi iteration matrix associated with $\mathscr{M}(\tilde{A}(\alpha))$ given by

$$|\tilde{B}(\alpha)| := \widetilde{D}(\alpha)^{-1}\big(|\tilde{L}(\alpha)| + \tilde{U}(\alpha)\big),$$

whose elements are all nonnegative and are given by the expressions

$$|\tilde{b}_{ij}(\alpha)| = \begin{cases} 0 & \text{if } i = j \in N, \\ -a_{ij} = b_{ij} & \text{if } i \in N\backslash N_2, \quad j \in N_1, \quad j \neq i, \\ \dfrac{a_{ij} - \alpha_i a_{ij}}{1 - \alpha_i a_{i1} a_{1i}} = \dfrac{(\alpha_i - 1) b_{ij}}{1 - \alpha_i b_{i1} b_{1i}} & \text{if } i \in N_2, \quad j = 1, \\ \dfrac{-a_{ij} + \alpha_i a_{i1} a_{1j}}{1 - \alpha_i a_{i1} a_{1i}} = \dfrac{b_{ij} + \alpha_i b_{i1} b_{1j}}{1 - \alpha_i b_{i1} b_{1i}} & \text{if } i \in N_2, \quad j \in N_1, \quad j \neq i. \end{cases} \tag{2.46}$$

Here we give a lemma which will be used in some of the proofs in the sequel.

Lemma 2.2. *Let a, b, c, d be constants with $|c| + |d| \neq 0$. Then*

$$\operatorname{sign}\left(\frac{\partial}{\partial z}\Big(\frac{az + b}{cz + d}\Big)\right) = \operatorname{sign}(ad - bc). \tag{2.47}$$

Applying Lemma 2.2 directly to the elements of $|\tilde{B}(\alpha)|$ given in (2.46) we have

$$\begin{aligned} &\operatorname{sign}\left(\frac{\partial|\tilde{b}_{i1}(\alpha)|}{\partial \alpha_i}\right) = \operatorname{sign}(b_{i1}(1 - b_{i1} b_{1i})) = 1,\ i \in N_2, \\ &\operatorname{sign}\left(\frac{\partial|\tilde{b}_{ij}(\alpha)|}{\partial \alpha_i}\right) = \operatorname{sign}(b_{i1}(b_{1j} + b_{1i} b_{ij})) = 1,\ i \in N_2, j \in N_1, j \neq i. \end{aligned} \tag{2.48}$$

From (2.48) we have some results which are stated and proved in the statements below.

Theorem 2.3. (a) *For any two distinct* $\alpha, \alpha' \in \mathbb{R}^{n-1}$, *with components* $\alpha_i, \alpha_i' \in I_i,\ i \in N_2$, *such that* $\alpha_i \leqslant \alpha_i'$, *with* $[1\ \ 1\ \ \cdots\ \ 1]^{\mathrm{T}} \leqslant \alpha = [\alpha_2\ \ \alpha_3\ \ \cdots\ \ \alpha_n]^{\mathrm{T}} \leqslant \alpha' = [\alpha_2'\ \ \alpha_3'\ \ \cdots\ \ \alpha_n']^{\mathrm{T}}$, *and where* $\alpha_i = \alpha_i' = 1,\ i \in N_1 \backslash N_2$, *we have*

$$\rho\big(\tilde{B}(1)\big) \equiv \rho\big(|\tilde{B}(1)|\big) \leqslant \rho\big(|\tilde{B}(\alpha)|\big) \leqslant \rho\big(|\tilde{B}(\alpha')|\big) < 1. \tag{2.49}$$

(b) *Moreover, except for some very special cases, described in the proof (see also* [4]), *increasing a certain* $\alpha_i \in I_i,\ \ i \in N_2$, *from* 1 *onwards, with all the other* α_i*'s remaining fixed, there exists a value of it, denoted by* $\hat{\alpha}_i$, *strictly to the right of* I_i *and less than* $1/(a_{i1}a_{1i})$, *such that*

$$\rho\big(\big|\tilde{B}\big([\alpha_2\ \ \alpha_3\ \ \cdots\ \ \hat{\alpha}_i\ \ \cdots\ \ \alpha_n]^{\mathrm{T}}\big)\big|\big) = 1. \tag{2.50}$$

(c) *If A is irreducible and* α *and* α' *are as in part* (a), *with* $\alpha_i, \alpha_i' \in I_i,\ i \in N_2$, *then the inequalities in* (2.49) *are strict.*

Proof. (a) By (2.48) the two leftmost inequalities in (2.49) come from the fact that the spectral radius of a nonnegative matrix, $|\tilde{B}(\alpha)|$, does not decrease if any of its entries increases (Theorem 2.20 of [14]). The rightmost inequality holds because for the values of α_i's, $i \in N_2$, $\mathscr{M}(\tilde{A}(\alpha))$ is a nonsingular M-matrix and its associated Jacobi iteration matrix converges.

(b) According to equation (2.41) of Section 2.3 of [14], the *normal form* of $\mathscr{M}(\tilde{A}(\alpha))$, obtained when a suitable similarity permutation tranformation is applied to it, is block upper triangular with diagonal blocks irreducible or 1×1 matrices. The same block property is inherited by the normal form $|\tilde{B}^{(n)}(\alpha)|$ of $|\tilde{B}(\alpha)|$, except that the 1×1 diagonal blocks in $\mathscr{M}^{(n)}(\tilde{A}(\alpha))$ are replaced by zeros. (Note: The notation $C^{(n)}$ is used for the normal form of $C \in \mathbb{C}^{n,n}$.) Due to the permutation tranformation above, the rows of the nonzero a_{i1}'s in the normal form may differ from the original ones but the new positions of all a_{i1}'s will be in one column. If the new position of a certain a_{i1} is row-wise associated with a zero block of $|\tilde{B}^{(n)}(\alpha)|$ then increasing α_i will not affect the spectral radius of $|\tilde{B}(\alpha)|$. If, however, it is row-wise associated with any irreducible diagonal block then increasing $\alpha_i,\ i \in N_2$, from 1 to $1/(\alpha_{i1}\alpha_{1i})$, since $a_{1i} \neq 0$, the spectral radius of the corresponding diagonal block in $|\tilde{B}^{(n)}(\alpha)|$ will strictly increase from its present value ($\leqslant \rho(|\tilde{B}(\alpha)|)$) to ∞, because $\tilde{\alpha}_{ii}(\alpha)$ will tend to zero. Consequently, there will be a value of α_i, denoted by $\hat{\alpha}_i$, for which $\rho(|\tilde{B}(\alpha)|) = 1$. Obviously $\hat{\alpha}_i$ will be strictly to the right of the open interval I_i since for α_i equal to the right end of I_i, $\rho(|\tilde{B}(\alpha)|) \leqslant 1$.

If A is irreducible then it is in its normal form and the validity of the statement holds true.

(c) If A is irreducible so are $\tilde{A}(\alpha),\ \tilde{A}(\alpha')$ and $|\tilde{B}(\alpha)|,\ |\tilde{B}(\alpha')|$. Hence by Theorem 2.7 of [14] the strictness of the middle inequality holds. For the leftmost inequality we have from Theorem 2.1 that the $(n-1) \times (n-1)$ matrix of the right lower corner of $\tilde{B}(1)$, denoted by $\tilde{B}_1(1)$, is irreducible and has $\rho(\tilde{B}_1(1)) = \rho(\tilde{B}(1))$. The

$(n-1) \times (n-1)$ matrix of the right lower corner of the matrix $|\tilde{B}(\alpha)|$, $|\tilde{B}_1(\alpha)|$, is also irreducible and has at least one of its elements strictly greater than the corresponding one of $\tilde{B}_1(1)$. Since $\rho(\tilde{B}(1)) = \rho(\tilde{B}_1(1)) < \rho(|\tilde{B}_1(\alpha)|) \leqslant \rho(|\tilde{B}(\alpha)|)$, with the last inequality holding because $0 \leqslant |\hat{B}(\alpha)| \leqslant |\tilde{B}(\alpha)|$, where $|\hat{B}(\alpha)|$ is an $n \times n$ matrix with $|\hat{B}_1(\alpha)| = |\tilde{B}_1(\alpha)|$ and all its other elements zero. The previous series of inequalities in terms of spectral radii proves the strictness of the leftmost inequality in (2.49). □

Theorem 2.4. *Let $H(\mathscr{M}(\tilde{A}(\alpha)))$ denote the Gauss–Seidel iteration matrix associated with $\mathscr{M}(\tilde{A}(\alpha))$. Then*:

(a) *For any $\alpha \in \mathbb{R}^{n-1}$, with components $\alpha_i \in I_i$, $i \in N_2$, there hold*

$$\rho\big(H\big(\mathscr{M}(\tilde{A}(\alpha))\big)\big) \leqslant \rho\big(|\tilde{B}(\alpha)|\big) < 1. \tag{2.51}$$

(b) *For any two distinct $\alpha, \alpha' \in \mathbb{R}^{n-1}$, with components $\alpha_i, \alpha_i' \in I_i$, $i \in N_2$, such that $\alpha_i \leqslant \alpha_i'$, with $[1\ 1\ \cdots\ 1]^{\mathrm{T}} \leqslant \alpha = [\alpha_2\ \alpha_3\ \cdots\ \alpha_n]^{\mathrm{T}} \leqslant \alpha' = [\alpha_2'\ \alpha_3'\ \cdots\ \alpha_n']^{\mathrm{T}}$, and where $\alpha_i = \alpha_i' = 1$, $i \in N_1 \backslash N_2$, we have*

$$\rho(\widetilde{H}(1)) \equiv \rho\big(H\big(\mathscr{M}(\tilde{A}(1))\big)\big) \leqslant \rho\big(H\big(\mathscr{M}(\tilde{A}(\alpha))\big)\big) \leqslant \rho\big(H\big(\mathscr{M}(\tilde{A}(\alpha'))\big)\big) < 1. \tag{2.52}$$

(c) *Moreover, except for some very special cases as in Theorem* 2.3, *increasing a certain $\alpha_i \in I_i$, $i \in N_2$, from* 1 *onwards, with all the other α_i's remaining fixed, there exists a value of it, denoted by $\hat{\alpha}_i$, strictly to the right of I_i and strictly less than $1/(a_{i1}a_{1i})$, such that*

$$\rho\big(H\big(\mathscr{M}\big(\tilde{A}\big([\alpha_2\ \alpha_3\ \cdots\ \hat{\alpha}_i\ \cdots\ \alpha_n]^{\mathrm{T}}\big)\big)\big)\big) = 1. \tag{2.53}$$

(d) *If A is irreducible and α and α' are as in part* (b) *with $\alpha_i, \alpha_i' \in I_i \backslash \{1\}$, $i \in N_2$, then the inequalities in* (2.51) *and* (2.52) *are strict*.

Proof. (a) $\mathscr{M}(\tilde{A}(\alpha))$ is a nonsingular M-matrix and relationships in (2.51) result from the application of the Stein–Rosenberg Theorem to the matrix in question.

(b) The Gauss–Seidel iteration matrix associated with $\mathscr{M}(\tilde{A}(\alpha))$ can be written as

$$\begin{aligned} H\big(\mathscr{M}(\tilde{A}(\alpha))\big) &= \big(\widetilde{D}(\alpha) - |\tilde{L}(\alpha)|\big)^{-1}\tilde{U}(\alpha) \\ &= \big(I - \widetilde{D}^{-1}(\alpha)|\tilde{L}(\alpha)|\big)^{-1}\widetilde{D}^{-1}(\alpha)\tilde{U}(\alpha). \end{aligned} \tag{2.54}$$

However, the matrices $\widetilde{D}^{-1}(\alpha)|\tilde{L}(\alpha)|$ and $\widetilde{D}^{-1}(\alpha)\tilde{U}(\alpha)$ are the strictly lower and strictly upper triangular parts of the Jacobi iteration matrix associated with $\mathscr{M}(\tilde{A}(\alpha))$ which, in view of (2.46) and (2.48), has its elements nonnegative and nondecreasing functions of any of the α_i's, with $\alpha_i \in I_i,\ i \in N_2$. Setting $\overline{L}(\alpha) = \widetilde{D}^{-1}(\alpha)|\tilde{L}(\alpha)| \geqslant 0$ and $\overline{U}(\alpha) = \widetilde{D}^{-1}(\alpha)\tilde{U}(\alpha) \geqslant 0$ the expression in (2.54) can be written as

$$\begin{aligned} H\big(\mathscr{M}(\tilde{A}(\alpha))\big) &= \big(I - \overline{L}(\alpha)\big)^{-1}\overline{U}(\alpha) \\ &= \big(I + \overline{L}(\alpha) + \overline{L}^{2}(\alpha) + \cdots + \overline{L}^{n-1}(\alpha)\big)\overline{U}(\alpha) \geqslant 0. \end{aligned} \tag{2.55}$$

So, if any $\alpha_i,\ i \in N_2$, increases then so do the elements of $\overline{L}(\alpha)$ and $\overline{U}(\alpha)$ and therefore the elements of $H(\mathscr{M}(\tilde{A}(\alpha)))$. Consequently, the inequalities in (2.52) hold true. A reasoning similar to that in the corresponding case of part (b) of Theorem 2.3 concludes the proof.

(c) In view of the expression for $H(\mathscr{M}(\tilde{A}(\alpha)))$ in (2.55) an argumentation similar to that in part (b) of Theorem 2.3 proves our assertion.

(d) The proof for the strictness of the inequality in (2.51) is a consequence of the Stein–Rosenberg Theorem. The corresponding proof for the strictness of the inequalities in (2.52) follows the same lines as those in Theorems 2.1 and 2.3 and is omitted. □

2.2. *"Best" convergent Jacobi and Gauss–Seidel iterative schemes*

Out of the convergent Jacobi and Gauss–Seidel iterative schemes derived in the previous Section 2.1 and for the nonsingular M-matrices A considered in Cases I and II one may wonder which of the permissible $\alpha \in \mathbb{R}^{n-1}$ gives the fastest iterative schemes.

In Case I, for the general class of nonsingular M-matrices, the best α is given by Theorem 2.2. That is, out of all $\alpha \in \mathbb{R}^{n-1}$, $\alpha = [a_2\ \ a_3\ \ \cdots\ \ \alpha_n]^{\mathrm{T}} = [1\ \ 1\ \ \cdots\ \ 1]^{\mathrm{T}} \in \mathbb{R}^{n-1}$ gives both the best Jacobi and and the best Gauss–Seidel iterative schemes.

In Case II, by Theorems 2.3 and 2.4 we conclude that for the matrices A dealt with there if $\alpha_i \in I_i \cup \{1\},\ i \in N_2$, the "best" Jacobi and Gauss–Seidel type schemes, which are associated with the comparison matrix $\mathscr{M}(\tilde{A}(\alpha))$ of A, are the ones with $\alpha = [1\ \ 1\ \ \cdots\ \ 1]^{\mathrm{T}} \in \mathbb{R}^{n-1}$.

3. Generalizing Kohno et al.'s iterative schemes

The generalized (parametrized) preconditioner we use in this section is that in (1.6). Our analysis follows the steps of Section 2 and extends the existing theory in various directions. First, we consider the Jacobi and Jacobi type methods associated with the preconditioned matrix A, $\hat{A}(\alpha) = P_2(\alpha)A$, where $\alpha = [\alpha_1\, \alpha_2\, \cdots\, \alpha_{n-1}]^{\mathrm{T}} \in \mathbb{R}^{n-1},\ \alpha_i \geqslant 0,\ i = 1(1)n-1$, extending the main theory in [3]. Next, we consider the Gauss–Seidel and Gauss–Seidel type iterative methods associated with $\hat{A}(\alpha)$, extending and completing the theory developed in [3,8]. (Note: In [9], a more general case than the one we consider is examined and many results especially for the case

$\alpha_i \in [0, 1],\ i \in N_2$, are given to which the reader is referred.) Then, comparisons of the corresponding Jacobi and Gauss–Seidel type methods are made, regarding the spectral radii of their iteration matrices in case of convergence, and finally, the "best" of the schemes considered is presented. The sets of integers to be used and the *additional assumption* made in the previous section are redefined as:

$$N := \{1, 2, \ldots, n\}, \quad N_1 := N\backslash\{n\}, \quad N_2 := \{i \in N_1 : a_{i,i+1} \neq 0\}. \tag{3.1}$$

Additional assumption: We assume that there exists a pair of indices $i \in N_2$ and $j \in N_1$ such that $a_{i,i+1}a_{i+1,j} \neq 0$, so that at least one element of $\hat{A}(\alpha)$ is different from that of A.

3.1. Jacobi and Gauss–Seidel type iterative schemes

Applying the new preconditioner $P_2(\alpha)$ on (1.1) we obtain a linear system which looks precisely like the one in (2.3) with the elements $\hat{a}_{ij}(\alpha)$ of $\hat{A}(\alpha)$ being given by

$$\hat{a}_{ij}(\alpha) = \begin{cases} a_{ij} - \alpha_i a_{i,i+1}a_{i+1,j}, & i \in N_1, \quad j \in N_1\backslash\{i+1\}, \\ (1-\alpha_i)a_{i,i+1}, & i \in N_1, \quad j = i+1, \\ a_{nj}, & i = n, \quad j \in N. \end{cases} \tag{3.2}$$

Case I. $\alpha_i \in [0, 1],\ i \in N_2$. Defining the matrices

$$\begin{aligned} &\widehat{D}_\alpha := \mathrm{diag}(\alpha_1 a_{12}a_{21}, \ldots, \alpha_{n-1}a_{n-1,n}a_{n,n-1}, 0) \quad \text{and} \\ &S_2(\alpha)L := \hat{L}_\alpha + \widehat{D}_\alpha, \end{aligned} \tag{3.3}$$

where $\widehat{D}_\alpha$ and $\hat{L}_\alpha$ the diagonal and the strictly lower triangular components of $S_2(\alpha)L$, then from (3.2) and the preceding discussion, we have that the three matrices on the right hand side of the corresponding to (2.4) relationships are given by

$$\widehat{D}(\alpha) = I - \widehat{D}_\alpha, \quad \hat{L}(\alpha) = L + \hat{L}_\alpha, \quad \hat{U}(\alpha) = (I + S_2(\alpha))\,U - S_2(\alpha). \tag{3.4}$$

$U - S_2(\alpha)$ is nonnegative, the diagonal elements of $\widehat{D}(\alpha)$ are positive while $\hat{L}(\alpha)$ and $\hat{U}(\alpha)$ are nonnegative. In the sequel the following splittings will be considered:

$$\hat{A}(\alpha) = \begin{cases} M(\alpha) - N(\alpha) = (I + S_2(\alpha)) - (I + S_2(\alpha))(L + U), \\ M'(\alpha) - N'(\alpha) = I - (L + \hat{L}_\alpha + \widehat{D}_\alpha + (I + S_2(\alpha))U - S_2(\alpha)), \\ M''(\alpha) - N''(\alpha) = (I - \widehat{D}_\alpha) - (L + \hat{L}_\alpha + (I + S_2(\alpha))\,U - S_2(\alpha)), \end{cases} \tag{3.5}$$

to define the Jacobi and the Jacobi type iteration matrices associated with them. The corresponding splittings for the Gauss–Seidel and the Gauss–Seidel type matrices are not given but become indirectly obvious from the iteration matrices we are to consider.

Below, we give the Jacobi and the Jacobi type iteration matrices defined from the splittings (3.5) as well as the corresponding Gauss–Seidel and Gauss–Seidel type ones:

$$\begin{aligned} B &:= M^{-1}(\alpha)N(\alpha) = L + U, \\ \hat{B}'(\alpha) &:= M'^{-1}(\alpha)N'(\alpha) = L + \hat{L}_\alpha + \widehat{D}_\alpha + (I + S_2(\alpha))U - S_2(\alpha), \\ \hat{B}''(\alpha) &:= M''^{-1}(\alpha)N''(\alpha) = \left(I - \widehat{D}_\alpha\right)^{-1}(L + \hat{L}_\alpha + (I + S_2(\alpha))\, U - S_2(\alpha)), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} H &:= (I - L)^{-1}U, \\ \widehat{H}'(\alpha) &:= (I - L - \hat{L}_\alpha)^{-1}\left(\widehat{D}_\alpha + (I + S_2(\alpha))U - S_2(\alpha)\right), \\ \widehat{H}''(\alpha) &:= \left(I - \widehat{D}_\alpha - L - \hat{L}_\alpha\right)^{-1}((I + S_2(\alpha))U - S_2(\alpha)). \end{aligned} \tag{3.7}$$

Theorem 3.1. (a) *Under the assumptions and the notation so far, for any* $\alpha \in K_{n-1}$, *where* K_{n-1} *is the* $(n-1)$*-dimensional nonnegative cone, such that* $\alpha_i \in [0, 1],\ i \in N_2$, *there hold*:

There exists $y \in \mathbb{R}^n$, *with* $y \geqslant 0$, *such that*

$$\hat{B}'(\alpha)y \leqslant By, \tag{3.8}$$

$$\rho\left(\hat{B}''(\alpha)\right) \leqslant \rho\left(\hat{B}'(\alpha)\right) < 1, \tag{3.9}$$

$$\rho\left(\widehat{H}''(\alpha)\right) \leqslant \rho\left(\widehat{H}'(\alpha)\right) \leqslant \rho(H) < 1, \tag{3.10}$$

$$\rho\left(\widehat{H}''(\alpha)\right) \leqslant \rho\left(\hat{B}''(\alpha)\right), \quad \rho\left(\widehat{H}'(\alpha)\right) \leqslant \rho\left(\hat{B}'(\alpha)\right), \quad \rho(H) < \rho(B) < 1. \tag{3.11}$$

(*Notes*: (i) *Equalities in* (3.11) *hold if and only if* $\rho(B) = 0$. (ii) *In* [13] *it is proved that* (3.8) *implies* $\rho(\hat{B}'(\alpha)) \leqslant \rho(B)$.)

(b) *Suppose that A is irreducible. Then*:

(i) *For* $\alpha_i \in [0, 1),\ i \in N_2$, *provided that* $\alpha \neq 0$, *the matrices* $\hat{B}''(\alpha),\ \hat{B}'(\alpha)$ *and* B *are irreducible and all the inequalities in* (3.9)–(3.11) *are strict. Moreover, there holds*

$$\rho(\hat{B}'(\alpha)) \leqslant \rho(B). \tag{3.12}$$

(ii) *For* $\alpha_i = 1,\ \ i \in N_2$, *the matrices* $\hat{B}''(1),\ \hat{B}'(1)$ *and* B *are irreducible and all the inequalities in* (3.9)–(3.12) *are strict*.

Proof. (a) (3.8): The expressions of the nonnegative elements of the Jacobi type iteration matrix $\hat{B}'(\alpha)$ are the following:

$$\begin{cases} \hat{b}'_{ii}(\alpha) = \alpha_i a_{i,i+1}a_{i+1,i} & \\ \qquad\quad = \alpha_i b_{i,i+1}b_{i+1,i}, & i \in N_2, \\ \hat{b}'_{ii}(\alpha) = 0, & i \in N\backslash N_2, \\ \hat{b}'_{ij}(\alpha) = -a_{ij} = b_{ij}, & i \in N\backslash N_2,\quad j \in N_1,\quad j \neq i, \\ \hat{b}'_{i,i+1}(\alpha) = (\alpha_i - 1)a_{i,i+1} & \\ \qquad\qquad = (1-\alpha_i)b_{i,i+1}, & i \in N_2, \\ \hat{b}'_{ij}(\alpha) = \alpha_i a_{i,i+1}a_{i+1,j} - a_{ij} & \\ \qquad\quad = \alpha_i b_{i,i+1}b_{i+1,j} + b_{ij}, & i \in N_2,\quad j \in N_1,\quad j \neq i, i+1. \end{cases} \tag{3.13}$$

It is observed that (3.13) are exactly the same expressions as those in (2.18) with the index $i+1$ replacing 1. So, the proof of part (a) of the present theorem follows the same lines as that of Theorem 2.1 and also use of regular splittings is made.

(bi) For $\alpha_i \in [0,1)$, if A is irreducible then so is $\hat{A}(\alpha)$, since the nonzero structure of A is inherited by $\hat{A}(\alpha)$, as is seen from (3.13). The irreducibility of $\hat{A}(\alpha)$ and the way of proof in Theorem 2.1 guarantee that the inequalities in (3.9)–(3.12) are strict.

(bii) The irreducibility of $\hat{A}(1)$ from that of A is proved as follows: From (3.2) it is implied that if an element $a_{ij},\ j \neq i+1$, of A is nonzero then so is $\hat{\alpha}_{ij}(1)$ of $\hat{A}(1)$. If $a_{ij} = 0$ then there will be a path in the graph $\mathscr{G}(A)$ of A joining the node i with the node j. Let this path consist of the edges joining the consecutive nodes $i,\ i_1,\ \ldots,\ i_p,\ i_q,\ i_r,\ \ldots,\ i_s,\ j$. Then $a_{i,i_1}\cdots a_{i_p,i_q}a_{i_q,i_r}\cdots a_{i_s,j} \neq 0$. If none of the nodes of the path is the node $i+1$ then, because $\hat{a}_{i,i_1}(1)\cdots\hat{a}_{i_p,i_q}(1)\hat{a}_{i_q,i_r}(1)\cdots\hat{a}_{i_s,j}(1) \neq 0$, the same path will be present in the graph $\mathscr{G}(\hat{A}(1))$ of $\hat{A}(1)$. If, however, one of the nodes is the node $q = i+1$, then because $a_{i_p,i_q}a_{i_q,i_r} \neq 0$, it will be $\hat{a}_{i_p,i_r} \neq 0$, and therefore $\hat{a}_{i,i_1}(1)\cdots\hat{a}_{i_p,i_r}(1)\cdots\hat{a}_{i_s,j}(1) \neq 0$, implying that in $\mathscr{G}(\hat{A}(1))$, there is still a path joining the nodes i and j. Consequently, $\hat{A}(1)$ is irreducible. From this irreducibility that of the three Jacobi and Jacobi type matrices readily follows and from the latter follows the strictness of the inequalities in (3.9)–(3.11). □

Remark 3.1. It is pointed out that the relationships in (3.8), indirectly, (3.10), the rightmost of (3.11) and (3.12) can be found in [9] as special cases of more general ones in Theorems 3.1, 3.2 and especially in Corollaries 3.1–3.3.

A monotonicity result analogous to Theorem 2.2 is presented below.

Theorem 3.2. *Under the assumptions and the notation so far let*

$$[0\ \ 0\ \ \cdots\ \ 0]^{\mathrm{T}} \leqslant \alpha \leqslant \alpha' \leqslant [1\ \ 1\ \ \cdots\ \ 1]^{\mathrm{T}} \in K_{n-1}. \tag{3.14}$$

Then

$$\rho\big(\hat{B}(\alpha')\big) \leqslant \rho\big(\hat{B}(\alpha)\big) \quad \textit{and} \quad \rho\left(\widehat{H}(\alpha')\right) \leqslant \rho\left(\widehat{H}(\alpha)\right). \tag{3.15}$$

Proof. The proof is completely different from that of Theorem 2.2. First, we observe again that the Jacobi and the Gauss–Seidel iterative methods associated with the preconditioned matrix $\hat{A}(\alpha)$, using any parameter α as in (3.14), are no worse than the ones associated with the original unpreconditioned matrix A. It is also noted that $\widehat{D}^{-1}\hat{A}$ has the same Jacobi and Gauss–Seidel iteration matrices as $\hat{A}$. From (3.2), the elements of the former matrix, denoted by the same symbols as those of $\hat{A}$, are:

$$\begin{aligned}
&\hat{a}_{ii} = 1, \ \ i \in N, \quad \hat{a}_{nj} = a_{nj}, \ \ j \in N_1, \\
&\hat{a}_{i,i+1} = \frac{(1-\alpha_i)a_{i,i+1}}{1-\alpha_i a_{i,i+1}a_{i+1,i}}, \ \ i \in N_1, \\
&\hat{a}_{ij} = \frac{a_{ij} - \alpha_i a_{i,i+1}a_{i+1,j}}{1-\alpha_i a_{i,i+1}a_{i+1,i}}, \ \ i \in N_1, \ \ j \in N_1\backslash\{i\}.
\end{aligned} \tag{3.16}$$

The key observation now is that $P_2(\alpha) = I + S_2(\alpha)$ can be written as the product below:

$$\begin{aligned}
P(\alpha) &= I + S(\alpha) \\
&= (I + S_{n-1}(\alpha_{n-1}))(I + S_{n-2}(\alpha_{n-2}))\cdots(I + S_2(\alpha_2))(I + S_1(\alpha_1)),
\end{aligned}$$

where $S_i(\alpha_i)$, $i = 1(1)n-1$, has all its elements zero except its $(i, i+1)$st which is $-\alpha_i a_{i,i+1}$. Then, we define the sequence of the matrices $\hat{A}_k(\alpha)$ by

$$\hat{A}_k(\alpha) = (I + S_k(\alpha_k))\hat{A}_{k-1}(\alpha), \quad k = 1(1)n-1, \quad \hat{A}_0(\alpha) = A. \tag{3.17}$$

Since $\hat{A}_k(\alpha)$ is the matrix obtained by applying the preconditioner $I + S_k(\alpha_k)$ to $\hat{A}_{k-1}(\alpha)$, then Kohno et al.'s preconditioned matrix is given by $\hat{A}(\alpha) = \hat{A}_{n-1}(\alpha)$. Therefore, by using an analogous notation, relations (3.15), that are to be proved, are rewritten as

$$\rho\big(\hat{B}_{n-1}(\alpha')\big) \leqslant \rho\big(\hat{B}_{n-1}(\alpha)\big) \quad \text{and} \quad \rho\big(\widehat{H}_{n-1}(\alpha')\big) \leqslant \rho\big(\widehat{H}_{n-1}(\alpha)\big). \tag{3.18}$$

To prove (3.18), we will prove the more general properties

$$\rho\big(\hat{B}_k(\alpha')\big) \leqslant \rho\big(\hat{B}_k(\alpha)\big) \quad \text{and} \quad \rho\big(\widehat{H}_k(\alpha')\big) \leqslant \rho\big(\widehat{H}_k(\alpha)\big), \tag{3.19}$$

by induction on k. First we consider the vector $\beta \in K_{n-1}$ whose components are defined by

$$\beta_i = 0, \quad \text{if } \alpha_i = 1 \quad \text{and} \quad \beta_i = \frac{\alpha_i' - \alpha_i}{1 - \alpha_i} \ (\in [0, 1]), \quad \text{if } \alpha_i \neq 1.$$

It is easy to see that the properties are true for $k = 1$ since $\hat{A}_1(\alpha')$ is the preconditioned matrix of $\hat{A}_1(\alpha)$ using as preconditioner $I + S_1(\beta_1)$. We assume that the properties (3.19) are true for $k = i \ (< n - 1)$ and we prove them for $k = i + 1$. The matrices $\hat{A}_i(\alpha)$ and $\hat{A}_i(\alpha')$ differ only in their first i rows, consequently in the same rows will differ the associated Jacobi iteration matrices $\hat{B}_i(\alpha)$ and $\hat{B}_i(\alpha')$. Applying the preconditioner $I + S_{i+1}(\alpha_{i+1})$ to both of the above matrices, only their $(i + 1)$st rows change but remain equal to each other. Recall Theorem 2.9 of [14] saying that: "For any irreducible nonnegative matrix A there holds

$$\sup_{x>0} \left\{ \min_{1 \leqslant i \leqslant n} \frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right\} = \rho(A) = \inf_{x>0} \left\{ \max_{1 \leqslant i \leqslant n} \frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right\}." \tag{3.20}$$

Applying this theorem to compare the spectral radii of the associated Jacobi iteration matrices we have that: If the minimum (or maximum) corresponds to any of the unchanged rows, then the same inequality holds for the spectral radii as in the unpreconditioned case. If the minimum (or maximum) corresponds to the $(i + 1)$st row for both matrices, then we have equal spectral radii. For both cases the conclusion is then that the Jacobi method in the second case is no worse than the one in the first case. We precondition the matrix $(I + S_{i+1}(\alpha_{i+1}))\hat{A}_i(\alpha')$ using as preconditioner the matrix $I + S_{i+1}(\beta_{i+1})$. It is easily checked that the matrix $\hat{A}_{i+1}(\alpha')$ is obtained. So, the associated Jacobi method is no worse than the previous one which is no worse than the Jacobi method associated with the matrix $\hat{A}_{i+1}(\alpha)$ and the proof for the Jacobi method is complete. The corresponding proof for the Gauss–Seidel method is given in an analogous way by using the fact that: If $\rho(H) = \rho\left((I - L)^{-1}U\right)$ is the spectral radius of the Gauss–Seidel iteration matrix, when $L, U \geqslant 0$, then it is also the spectral radius of the matrix $\rho(H)L + U$. The proof of this fact is based again on (3.20). The case of A being reducible is treated as a limiting case as this was done in Lemma 2.1. □

Case II. $\alpha_i > 1, \ i \in N_2$. In this case some of the elements of the Jacobi matrix $\hat{B}(\alpha)$ can be negative and this forces us to restrict the class of M-matrices we are studying to those which are SDD or IDD ones, as this was done in Section 2. Adopting similar notation and definitions, where a "hat" is used for a "tilde", analogously to (2.35)–(2.36) we set

$$\begin{aligned} &\hat{p}_i = a_{i,i+1}a_{i+1,i} \geqslant 0, \quad \hat{q}_i = a_{i,i+1} \sum_{j=1}^{i-1} a_{i+1,j} \geqslant 0, \\ &\hat{r}_i = a_{i,i+1} \sum_{j=i+1}^{n} a_{i+1,j} \leqslant 0, \end{aligned} \tag{3.21}$$

and so we have

$$\hat{p}_i + \hat{q}_i + \hat{r}_i = a_{i,i+1} \sum_{j=1}^{n} a_{i+1,j}$$
$$= a_{i,i+1}(1 - l_{i+1} - u_{i+1}) \leqslant 0, \quad i \in N_2. \tag{3.22}$$

For the Jacobi, $\hat{B}(\alpha)$, and the Gauss–Seidel, $\widehat{H}(\alpha)$, iteration matrices to converge we consider sufficient conditions analogous to those in (2.37). As in Section 2, the values, the α_i's can take, must not destroy the positivity of the diagonal elements $\hat{a}_{ii}(\alpha)$ and must preserve for each row of $\hat{A}(\alpha)$, at least, the inequalities that the corresponding row of A satisfies. Comparing (2.5) with (3.2) and (2.35)–(2.36) with (3.21)–(3.22) and using the corresponding to (2.37), one can realize that an analysis identical to that in Case II of Section 2 leads to the same expressions and relationships as in (2.38)–(2.44) except that the index 1 there is replaced by $i + 1$. Therefore, the intervals from which the α_i's, $i \in N_2$, can take values are

$$\alpha_i \in I_i := \left(1, \frac{1 - l_i - u_i - 2a_{i,i+1}}{(-a_{i,i+1})(1 + l_{i+1} + u_{i+1})}\right), \quad i \in N_2. \tag{3.23}$$

Note: In [8] it is said that α_i's $\geqslant 1, \ i \in N_2$, exist that make the corresponding $\widehat{u}_i(\alpha_i)$'s zero. However, this can not happen, as can be proved, except for α_{n-1} when $\alpha_{n-1} = 1$.

In view of (3.23) statements analogous to Theorems 2.3 and 2.4 can be stated and proved for the Jacobi, $B(\mathscr{M}(\hat{A}(\alpha)))$, and the Gauss–Seidel, $H(\mathscr{M}(\hat{A}(\alpha)))$, matrices associated with $\mathscr{M}(\hat{A}(\alpha))$. We only state them since their proofs are quite analogous to the previous ones and can be omitted. (Note: The interested reader is referred to [4].)

Theorem 3.3. (a) *For any two distinct* $\alpha, \alpha' \in \mathbb{R}^{n-1}$, *with components* $\alpha_i, \alpha'_i \in I_i$, $i \in N_2$, *such that* $\alpha_i \leqslant \alpha'_i$, *with* $[1 \ 1 \ \cdots \ 1]^{\mathrm{T}} \leqslant \alpha = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_{n-1}]^{\mathrm{T}} \leqslant \alpha' = [\alpha'_1 \ \alpha'_2 \ \cdots \ \alpha'_{n-1}]^{\mathrm{T}}$, *and where* $\alpha_i = \alpha'_i = 1, \ i \in N_1 \backslash N_2$, *we have*

$$\rho(\hat{B}(1)) \equiv \rho(|\hat{B}(1)|) \leqslant \rho(|\hat{B}(\alpha)|) \leqslant \rho(|\hat{B}(\alpha')|) < 1. \tag{3.24}$$

(b) *Moreover, except for some special cases, as in Theorem* 2.3, *increasing a certain* $\alpha_i \in I_i, \ i \in N_2$, *from* 1 *onwards, with all the other* α_i*'s remaining fixed, there exists a value of it, denoted by* $\hat{\hat{\alpha}}_i$, *strictly to the right of* I_i *and less than* $1/(a_{i,i+1}a_{i+1,i})$, *such that*

$$\rho\left(\left|\hat{B}\left([\alpha_1 \ \alpha_2 \ \cdots \ \hat{\hat{\alpha}}_i \ \cdots \ \alpha_{n-1}]^{\mathrm{T}}\right)\right|\right) = 1. \tag{3.25}$$

(c) *If* A *is irreducible and* α *and* α' *are as in part* (a), *with* $\alpha_i, \alpha'_i \in I_i, \ i \in N_2$, *then the inequalities in* (3.24) *are strict.*

Theorem 3.4. *Let $H(\mathscr{M}(\hat{A}(\alpha)))$ denote the Gauss–Seidel iteration matrix associated with $\mathscr{M}(\hat{A}(\alpha))$. Then*:

(a) *For any $\alpha \in \mathbb{R}^{n-1}$, with components $\alpha_i \in I_i,\ i \in N_2$, there hold*

$$\rho\big(H\big(\mathscr{M}(\hat{A}(\alpha))\big)\big) \leqslant \rho\big(|\hat{B}(\alpha)|\big) < 1. \tag{3.26}$$

(b) *For any two distinct $\alpha, \alpha' \in \mathbb{R}^{n-1}$, with components $\alpha_i, \alpha_i' \in I_i,\ i \in N_2$, such that $\alpha_i \leqslant \alpha_i'$, with $[1\ \ 1\ \ \cdots\ \ 1]^{\mathrm{T}} \leqslant \alpha = [\alpha_1\ \ \alpha_2\ \ \cdots\ \ \alpha_{n-1}]^{\mathrm{T}} \leqslant \alpha' = [\alpha_1'\ \ \alpha_2'\ \ \cdots\ \ \alpha_{n-1}']^{\mathrm{T}}$, and where $\alpha_i = \alpha_i' = 1,\ i \in N_1\backslash N_2$, we have*

$$\rho\big(\widehat{H}(1)\big) \equiv \rho\big(H\big(\mathscr{M}(\hat{A}(1))\big)\big) \leqslant \rho\big(H\big(\mathscr{M}(\hat{A}(\alpha))\big)\big) \leqslant \rho\big(H\big(\mathscr{M}(\hat{A}(\alpha'))\big)\big) < 1. \tag{3.27}$$

(c) *Moreover, except for some special cases, as in Theorem* 2.4, *increasing a certain $\alpha_i \in I_i,\ i \in N_2$, from* 1 *onwards, with all the other α_i's remaining fixed, there exists a value of it, denoted by $\hat{\hat{\alpha}}_i$, strictly to the right of the interval I_i and less than $1/(a_{i,i+1}a_{i+1,i})$, such that*

$$\rho\big(H\big(\mathscr{M}\big(\hat{A}\big([\alpha_1\ \ \alpha_2\ \ \cdots\ \ \hat{\hat{\alpha}}_i\ \ \cdots\ \ \alpha_{n-1}]^{\mathrm{T}}\big)\big)\big)\big) = 1. \tag{3.28}$$

(d) *If A is irreducible and α and α' are as in part* (b) *with $\alpha_i, \alpha_i' \in I_i\backslash\{1\},\ i \in N_2$, then the inequalities in* (3.26) *and* (3.27) *are strict*.

3.2. "Best" convergent Jacobi and Gauss–Seidel schemes

In Case I, Theorem 3.2 suggests that out of all $\alpha \in \mathbb{R}^{n-1}$, $\alpha = [a_2\ \ a_3\ \ \cdots\ \ \alpha_n]^{\mathrm{T}} = [1\ \ 1\ \ \cdots\ \ 1]^{\mathrm{T}} \in \mathbb{R}^{n-1}$ gives both the best Jacobi and and the best Gauss–Seidel iterative schemes for the entire class of nonsingular M-matrices A.

In Case II, and for the SDD and IDD M-matrices considered, in view of Theorems 3.3 and 3.4, we may accept, as in Section 2, that for both the Jacobi and the Gauss–Seidel methods the "best" α is, $\alpha = [1\ \ 1\ \ \cdots\ \ 1]^{\mathrm{T}} \in \mathbb{R}^{n-1}$.

4. Numerical examples

For over 10,000 randomly generated nonsingular M-matrices for $n = 5$–100 we determined the spectral radii of the iteration matrices of Jacobi, Gauss–Seidel as well as those of the corresponding iteration matrices after the application of the Milaszewicz's preconditioner, the Gunawardena et al.'s one and, also, the successive application of the two preconditioners. Below we present four representative example matrices for which the spectral radii of the corresponding iteration matrices considered are given in the subsequent tables. In the tables J and GS denote Jacobi and Gauss–Seidel type methods while the indices M and G denote that Milaszewicz's preconditioner and Gunawardena et al.'s were used, respectively. $M - G$

means that the application of Milaszewicz's preconditioner was used first followed by Gunawardena et al.'s, while $G-M$ means the reverse situation.

$$A_1 = \begin{bmatrix} 1.00000 & -0.00580 & -0.19350 & -0.25471 & -0.03885 \\ -0.28424 & 1.00000 & -0.16748 & -0.21780 & -0.21577 \\ -0.24764 & -0.26973 & 1.00000 & -0.18723 & -0.08949 \\ -0.13880 & -0.01165 & -0.25120 & 1.00000 & -0.13236 \\ -0.25809 & -0.08162 & -0.13940 & -0.04890 & 1.00000 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.00000 & -0.15359 & -0.24342 & -0.02303 & -0.03363 \\ -0.01756 & 1.00000 & -0.00630 & -0.14703 & -0.18174 \\ -0.01087 & -0.03714 & 1.00000 & -0.25258 & -0.17673 \\ -0.12507 & -0.01414 & -0.07603 & 1.00000 & -0.14130 \\ -0.00515 & -0.24496 & -0.23477 & -0.27707 & 1.00000 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1.00000 & -0.27149 & -0.20650 & -0.02972 & -0.12557 \\ -0.12416 & 1.00000 & -0.18328 & -0.07729 & -0.25528 \\ -0.31163 & -0.02827 & 1.00000 & -0.15184 & -0.39463 \\ -0.12292 & -0.00477 & -0.23299 & 1.00000 & -0.20115 \\ -0.37067 & -0.09086 & -0.20368 & -0.30835 & 1.00000 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1.00000 & -0.23661 & -0.37369 & -0.25833 & -0.05480 \\ -0.13602 & 1.00000 & -0.10578 & -0.38675 & -0.32750 \\ -0.12569 & -0.01525 & 1.00000 & -0.26597 & -0.17207 \\ -0.14603 & -0.18344 & -0.34914 & 1.00000 & -0.35613 \\ -0.15730 & -0.34795 & -0.09515 & -0.00397 & 1.00000 \end{bmatrix}.$$

Matrix	$\rho(J)$	$\rho(J_M)$	$\rho(J_G)$	$\rho(J_{G-M})$	$\rho(J_{M-G})$
A_1	0.629054	0.553502	0.584773	0.482347	0.460060
A_2	0.484223	0.460575	0.418960	0.391340	0.393935

Matrix	$\rho(GS)$	$\rho(GS_M)$	$\rho(GS_G)$	$\rho(GS_{G-M})$	$\rho(GS_{M-G})$
A_3	0.603046	0.480367	0.497869	0.340877	0.351696
A_4	0.684691	0.622791	0.568660	0.491844	0.490150

From all the examples we run, we can make the following observations: When each of the two preconditioners is used alone in connection with the Jacobi method in approximately 50% of the cases Gunawardena et al.'s (G) preconditioner gave better results than Milaszewicz's (M). However, when they were used with the

Gauss–Seidel method G preconditioner gave better results in approximately 70% of the cases. The successive application of the two preconditioners, as one might have expected from the theory developed, always gave better results than the application of either one of them. For the Jacobi method the application of the two preconditioners in the order $M - G$ gives better results than in the order $G - M$ in approximately 65% of the cases, while for the Gauss–Seidel method, in more than 90% of the cases the $M - G$ preconditioner gives better results. The percentages given seem to increase in almost all the cases as the order of the matrix increases from 5 to 100.

5. Concluding remarks and discussion

As one may have noticed from our analysis many questions are raised directly or indirectly. For example: Can the intervals of convergence for the α_i's be extended further for the entire class of nonsingular M-matrices? Can "optimal" values for the α's be obtained theoretically? Can one prove theoretically some of the "facts" that the numerical evidence provides? Can the two preconditioners be exploited further without increasing the computational complexity of the problem solved? Can these or similar preconditioners be used efficiently with other classes of matrices? These and many other issues have been studied and partial answers to some of them have already been given.

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