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Towards the determination of the optimal p-cyclic SSOR

A. Hadjidimos [a,*,1], D. Noutsos [b], M. Tzoumas [b]

[a] *Computer Science Department, Purdue University, West Lafafayette, IN 47907, United States*
[b] *Department of Mathematics, University of Ioannina, GR-451 10 Ioannina, Greece*

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Abstract

Suppose that $A \in \mathbb{C}^{n,n}$ is a block p-cyclic consistently ordered matrix and let B and S_ω denote the block Jacobi and the block symmetric successive overrelaxation (SSOR) iteration matrices associated with A, respectively. Extending previous work by Hadjidimos and Neumann, the present authors have determined the exact regions of convergence of the SSOR method in the $(\rho(B), \omega)$-plane, for any $p \geqslant 3$, under the further assumption that the eigenvalues of B^p are real of the same sign. In the present work the investigation goes on further, several questions are raised and among others the problem of the determination of the optimal regions of convergence in the spirit of Niethammer and Varga as well as that of the optimal relaxation factor are examined. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

We are given the nonsingular linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n,n}$ and $x, b \in \mathbb{C}^n$. Suppose that A is written in the $p \times p$ block form

$$A = D(I - L - U) \tag{1.2}$$

with D a $p \times p$ invertible block diagonal, $p \geqslant 3$, and L and U block strictly lower and strictly upper triangular matrices, respectively. Suppose also that for the solution of (1.1) and (1.2) the symmetric

* Correspondence address: Department of Mathematics, University of Crete, GR-714 09 Heraklion, Greece.
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successive overrelaxation (SSOR) iterative method (see, e.g., [25, 29, 1])

$$\begin{aligned} x^{(m+1/2)} &= (I-\omega L)^{-1}[(1-\omega)I+\omega U]x^{(m)}+\omega(I-\omega L)^{-1}D^{-1}b, \\ x^{(m+1)} &= (I-\omega U)^{-1}[(1-\omega)I+\omega L]x^{(m+1/2)}+\omega(I-\omega U)^{-1}D^{-1}b, \quad m=1,2,\ldots, \end{aligned} \tag{1.3}$$

where $x^{(0)} \in \mathbb{C}^n$ arbitrary and $\omega \in (0,2)$ the relaxation factor, is to be used. The block SSOR iteration matrix, associated with A, relative to its block partitioning, is given by

$$S_\omega := (I-\omega U)^{-1}[(1-\omega)I+\omega L](I-\omega L)^{-1}[(1-\omega)I+\omega U]. \tag{1.4}$$

Let $B := L+U$ be the block Jacobi matrix associated with A. If A is block p-cyclic consistently ordered then we may assume that B has the block form

$$B = \begin{bmatrix} 0 & 0 & 0 \ldots & 0 & B_1 \\ B_2 & 0 & 0 \ldots & 0 & 0 \\ 0 & B_3 & 0 \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & 0 \ldots & B_{p-1} & 0 \end{bmatrix}. \tag{1.5}$$

As is known the eigenvalues μ of B (or of B^{T}) and λ of S_ω satisfy the equation, discovered by Varga, et al. [26],

$$[\lambda-(1-\omega)^2]^p = \lambda(\lambda+1-\omega)^{p-2}(2-\omega)^2\omega^p\mu^p \tag{1.6}$$

which generalized the corresponding relationship for $p=2$ (see [2, 18]).

For the study of the convergence properties of the SSOR method any information about the spectrum of B, $\sigma(B)$, may enable one to answer one or more of the following questions:

(i) For what pairs $(\rho(B),\omega)$ does (1.3) converge ($\rho(S_\omega)<1$) and for what pairs $(\rho(B),\omega)$ does (1.3) converge in the sense of [22] (namely that $\rho(S_\omega)<1/\eta$, for a given $\eta \geqslant 1$)?

(ii) What is the largest region, in the complex plane, that contains $\sigma(B)$ for which (1.3) converges in the sense of [22] as in (i) previously? and

(iii) What is the (optimal) value of ω that minimizes $\rho(S_\omega)$ for a given $\rho(B)$ and what is the (optimal) region in the complex plane that contains a given $\sigma(B)$?

Complete answers to the first part of question (i) have been given by Neumaier and Varga [19] for the entire class of H–matrices, by Hadjidimos and Neumann [9] for consistently ordered matrices, and [10] for the class of p-cyclic matrices, and by Hadjidimos et al. [12] for the class of p-cyclic matrices with $\sigma(B^p)$ real of the same sign. An answer to question (ii) was given by Galanis et al. [7] for $p=2$ only. Finally, an answer to the first part of question (iii), for $p=2$, was given by D'Sylva and Miles [2] and by Lynn [18] and, in a more general case, by Hadjidimos and Noutsos [11]. It seems that for $p \geqslant 3$ the problems in the second part of question (i) and in questions (ii) and (iii) have not been studied so far. On the other hand, the corresponding problems in the simpler case of the p-cyclic SOR method have been extensively studied. Here we mention some of the researchers in this area and refer to their works; Young [28] (see also [29]), Varga [24] (see also [25]), Kredell [17], Niethammer [20], Young and Eidson [30] (see also [29]), Niethammer and Varga [22], Niethammer et al. [21], Hadjidimos, et al. [8], Galanis et al. [4–6], Wild and Niethammer [27] Eiermann, et al. [3], Kontovasilis, et al. [16], Hadjidimos and Plemmons [14], Noutsos [23] and others.

In this work we try to answer the questions, raised previously, for the p-cyclic consistently ordered SSOR case for $p \geqslant 3$. The reader is reminded that some parts of the exact boundaries in [12] had to be determined computationally because of the nature of the equations involved. Since the regions in [12] correspond to $\eta = 1$ while the ones in this work correspond to any $\eta > 1$ we must expect that some of the analogous results here can be found only computationally.

In Section 2 of this work, the mapping that connects the spectra in (1.6) is studied to find the conditions under which it is conformal or not. In Section 3, the study of the (optimal) regions of convergence and of the (optimal) relaxation factor ω is made and results for the nonnegative case are obtained. For the nonpositive only the main statements are presented; the analysis can be found in [13]. In Section 4, the special case $p = 3$ is briefly treated, separately. In Section 5 numerical results to support the theory developed are presented while in Section 6 some (theoretically) open questions in the form of conjectures are addressed.

2. Conformal mappings of the SSOR spectral regions

We begin our analysis with the functional equation (1.6) which is rewritten as follows:

$$\mu^p = \frac{[\lambda - (1-\omega)^2]^p}{(2-\omega)^2\omega^p\lambda(\lambda + 1 - \omega)^{p-2}}. \tag{2.1}$$

Using the transformation $\phi = 1/\lambda$ substituting into (2.1) and setting $z = \mu^p$, we obtain the mapping

$$z:\ z(\phi) = \frac{[1 - (1-\omega)^2\phi]^p}{(2-\omega)^2\omega^p\phi[1 + (1-\omega)\phi]^{p-2}}. \tag{2.2}$$

Our objective is to find the smallest region in the complex plane containing $\mu^p \in \sigma(B^p)$ which has an image, through the mapping (2.2), in the exterior of the circle

$$\partial D_\eta := \{\phi : \phi = \eta e^{i\theta},\ \eta > 0,\ \theta \in [0, 2\pi)\}, \tag{2.3}$$

where D_η is the corresponding disk, or, since $\lambda = (1/\eta)e^{-i\theta}$, in the interior of the circle $\partial D_{1/\eta}$. Then, the spectral radius of the SSOR iteration matrix will be less than or equal to $1/\eta$ ($\rho(S_\omega) \leqslant 1/\eta$) with equality holding if and only if (*iff*) there is an eigenvalue of B^p on the boundary of the region to be found.

For the solution of this problem we study the mapping (2.2) as η increases continuously from the value 0. For $\eta = 0$ the circle ∂D_η is trivially the point 0 of the complex plane. So, (2.2) transforms the point 0 onto the point ∞ and the mapping is conformal. Due to the continuity of the mapping as η varies, it will be conformal for some $\eta > 0$ in the neighborhood of 0. As is known, a mapping like (2.2) is not conformal if there is a $\phi \in D_\eta$ such that $dz/d\phi = 0$. This suggests that the smallest value of η, for which $dz/d\phi = 0$ for some $\phi \in \partial D_\eta$, must be found. Considering ω constant and differentiating (2.2) with respect to (w.r.t.) ϕ, we can obtain after some simple manipulation that

$$\frac{dz}{d\phi} = 0 \iff F(\phi) := (1-\omega)^3\phi^2 + (p-1)(1-\omega)(2-\omega)\phi + 1 = 0. \tag{2.4}$$

By considering the discriminant of the quadratic in (2.4) it is readily checked that $F(\phi)$ has only real zeros. Since $\phi = \eta e^{i\theta}$, the possible (real) zeros of (2.4) must correspond to $\theta = 0$ and $\theta = \pi$. Therefore, we have to distinguish two cases which are studied in the sequel.

Case 1: For $\theta = 0$, (2.4) gives

$$(1-\omega)^3\eta^2 + (p-1)(1-\omega)(2-\omega)\eta + 1 = 0 \tag{2.5}$$

and our problem turns out to be the determination of the smallest positive root of (2.5). Obviously, $\omega \neq 1$ because for $\omega = 1$, (2.5) cannot hold. The two roots of (2.5) are given by

$$\eta_{+,-} = \frac{-(p-1)(1-\omega)(2-\omega) \pm |1-\omega|\left[(p-1)^2(2-\omega)^2 + 4(\omega-1)\right]^{1/2}}{2(1-\omega)^3}. \tag{2.6}$$

For $\omega < 1$, $\eta_{+,-}$ are negative and the mapping is conformal. For $\omega > 1$, (2.6) gives

$$\eta_{-} = \frac{-(p-1)(2-\omega) + \left[(p-1)^2(2-\omega)^2 + 4(\omega-1)\right]^{1/2}}{2(1-\omega)^2}. \tag{2.7}$$

So, for $\eta \leqslant \eta_-$ the mapping (2.2) is conformal. For $\eta > \eta_-$ it is not and the image of the circle ∂D_η of (2.2) has an intersection (double) point on the real axis.

Case 2: For $\theta = \pi$, we have

$$(1-\omega)^3\eta^2 - (p-1)(1-\omega)(2-\omega)\eta + 1 = 0. \tag{2.8}$$

The two roots of (2.8) are given by

$$\eta_{+,-} = \frac{(p-1)(1-\omega)(2-\omega) \pm |1-\omega|\left[(p-1)^2(2-\omega)^2 + 4(\omega-1)\right]^{1/2}}{2(1-\omega)^3}. \tag{2.9}$$

For $\omega < 1$, the smallest positive root of (2.8) is

$$\eta_{-} = \frac{(p-1)(2-\omega) - \left[(p-1)^2(2-\omega)^2 + 4(\omega-1)\right]^{1/2}}{2(1-\omega)^2}. \tag{2.10}$$

The value $\eta = \eta_-$ just found is the only value of η at which the mapping (2.2) ceases to be conformal. This is because, as we saw before, for $\theta = 0$ the mapping is always conformal. For $\omega > 1$, the positive root of (2.8) is

$$\eta_{-} = \frac{(p-1)(2-\omega) + \left[(p-1)^2(2-\omega)^2 + 4(\omega-1)\right]^{1/2}}{2(1-\omega)^2}. \tag{2.11}$$

To find the value of η at which the mapping ceases to be conformal we compare the value $\eta = \eta_-$ in (2.11) with that given in (2.7) for $\theta = 0$. Obviously, the smallest of the two η's is the one corresponding to $\theta = 0$. We note that η_- in (2.11) is greater than $1/(1-\omega)^2$ and since for $\eta = 1/(1-\omega)^2$ and $\theta = 0$, z in (2.2) becomes zero it is implied that there exists no convergence region for the SSOR method for $\eta > 1/(1-\omega)^2$. This is not unexpected since $\rho(S_\omega) \geqslant (1-\omega)^2$ holds (see, e.g., [29]). Therefore, η_- in (2.11) is of no interest.

The analysis above leads to the following conclusions.

Theorem 2.1. *The transformation* (2.2) *maps the circle* ∂D_η *in* (2.3) *into a closed curve* C_p *in the complex plane. For* $\omega > 1$, *this mapping is conformal for all*

$$\eta \in (0, \hat{\eta}], \quad \hat{\eta} = \frac{-(p-1)(2-\omega) + [(p-1)^2(2-\omega)^2 + 4(\omega-1)]^{1/2}}{2(1-\omega)^2} \quad \left(< \frac{1}{(1-\omega)^2}\right) \tag{2.12}$$

and is not conformal otherwise, while for $\omega < 1$, *the mapping is conformal for all*

$$\eta \in (0, \tilde{\eta}], \quad \tilde{\eta} = \frac{(p-1)(2-\omega) - [(p-1)^2(2-\omega)^2 + 4(\omega-1)]^{1/2}}{2(1-\omega)^2} \quad \left(< \frac{1}{(1-\omega)^2}\right) \tag{2.13}$$

and is not conformal otherwise.

Corollary 2.2. *For the values of* η *of Theorem* 2.1 *for which the mapping* (2.2) *is conformal,* (2.2) *maps the interior of the disk* $D_\eta = \{\phi : |\phi| \leqslant \eta\}$ *onto the exterior of the closed simple curve* C_p.

Remark. The curve C_p is symmetric w.r.t. the real axis.

3. Optimal regions of convergence

The analysis in the previous section provides us with the main tool for the study of the (optimal) convergence properties of the SSOR iterative method. Since $1/\eta$ is the spectral radius of the SSOR iteration matrix, to have convergence, $\eta > 1$ must hold.

For $\omega = 1$ (Aitken's method) the curve C_p is the circle $(1/\eta)e^{-i\theta}$ and the mapping is conformal for all η. So, we distinguish two cases, depending on whether ω is greater or less than 1.

Case 1: $\omega \in (1,2)$. We take a certain $\omega \in (1,2)$ and increase η continuously from $\eta = 0$. For $\eta = 0$, (2.2) maps the point 0 of the complex plane onto ∞. For $0 < \eta < \hat{\eta}$, it maps the disk D_η onto the exterior of the curve C_p and the mapping is conformal. C_p is then a simple closed curve containing the point 0 in its interior. Moreover, for $\eta > 1$, the interior of C_p, let Ω denote it, is such that if $\sigma(B^p) \in \Omega$ then the associated SSOR method will converge with an asymptotic convergence factor $\rho(S_\omega)$ ($\leqslant 1/\eta$). Obviously, the larger η is the smaller the region Ω is. The region of convergence Ω is shown in Fig. 1(a) for $p = 4$, $\omega = 1.2$ and $\eta = 1.1$.

$\eta = \hat{\eta}$ is the largest value of η for which the mapping is conformal. Since $dz/d\phi|_{\phi=\hat{\eta}} = 0$, C_p has a cusp at $\theta = 0$. The corresponding convergence region Ω is shown in Fig. 1(b) for $p = 4$, $\omega = 1.2$, where from (2.12), $\hat{\eta} \approx 2.01562$.

For $\hat{\eta} < \eta < 1/(\omega - 1)$, the mapping is not conformal. The analysis shows that there is an intersection point of C_p on the positive real semiaxis and the convergence region Ω is the subregion formed by C_p that contains the point 0 in its interior and has its images in the exterior of the circle $\eta e^{i\theta}$. Due to the nature of the equations involved the intersection point for a specific pair ω and η can be found only computationally. Fig. 1(c) depicts the case $p = 4$, $\omega = 1.2$ and $\eta = 2.4$ and shows C_p, the intersection point and the region Ω which is the one on the left of the two subregions formed by C_p.

For $\eta = 1/(\omega - 1)$, the same conclusions hold except that the point of C_p corresponding to $\theta = 0$ goes to ∞. So, C_p is not a closed curve.

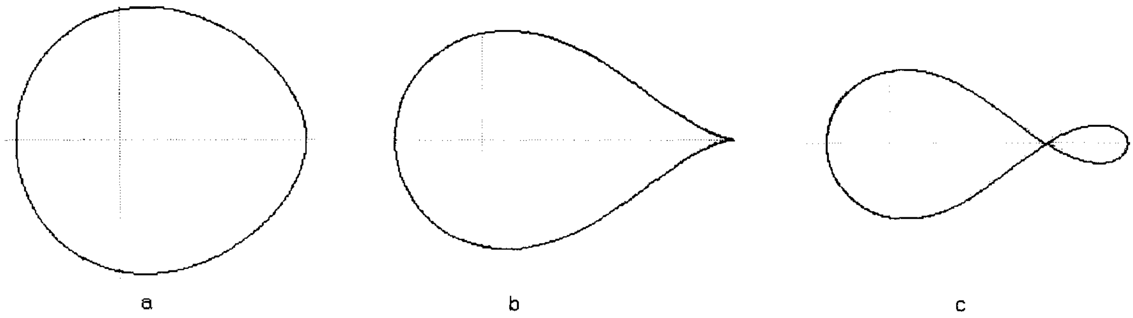

Fig. 1. Regions of convergence.

For $1/(\omega-1) < \eta < 1/(\omega-1)^2$, the point of C_p for $\theta = 0$ lies on the negative real semiaxis for odd p and on the positive real semiaxis for even p. The analysis shows that the curve C_p is much more complicated in shape. For some η's we have more than one intersection (double) points of the curve C_p with the real semiaxis and more intersection (double) points of C_p with itself. Due to the continuity of the mapping, the convergence region Ω is the smallest subregion formed by C_p that contains the point 0 in its interior.

For $\eta = 1/(\omega-1)^2$, the region of convergence reduces to the point 0 of the complex plane.

For $\eta > 1/(\omega-1)^2$, there exists *no* region of convergence for the SSOR method.

Case 2: $\omega \in (0, 1)$. Results analogous to the ones in the previous Case 1 can be obtained. The main differences are that $\tilde{\eta}$ takes the place of $\hat{\eta}$, $1/(1-\omega)$ replaces $1/(\omega-1)$, in the various intervals considered, and the roles of the positive and negative real semiaxes are interchanged.

Based on the analysis of this section we can state the following theorem.

Theorem 3.1. *Let $\eta > 1$ be given and let Ω be the region defined in the previous analysis for a given $\omega \in (1 - 1/\sqrt{\eta}, 1 + 1/\sqrt{\eta})$. Let also that $\sigma(B^p) \subset \Omega$. Then, the associated SSOR method converges with a spectral radius $\rho(S_\omega) \leqslant 1/\eta$. In the last relationship equality holds* iff *at least one element of $\sigma(B^p)$ lies on the boundary $\partial\Omega$ of Ω.*

The analysis so far gives answers to the second part of question (i) as well as to question (ii) of the Introduction. In the following two subsections we shall try to answer questions (i) and (iii) under the assumption that the spectra $\sigma(B^p)$ are real of the same sign.

3.1. Nonnegative case

In [12] we determined the regions of convergence of the SSOR method ($\rho(S_\omega) < 1$) in the $(\rho(B), \omega)$-plane for nonnegative and nonpositive spectra $\sigma(B^p)$ for all $p \geqslant 3$. Here we will try to determine regions in the $(\rho(B^p), \omega)$-plane such that $\rho(S_\omega) \leqslant 1/\eta$ for a given $\eta > 1$ and also, whenever possible, optimal values of the relaxation factor ω for a given $\rho(B)$.

For a given $\eta > 1$ we study the transformation (2.2) for all values of $\omega \in (0, 2)$. More specifically, in the present case we must determine the closest to the origin O(0, 0) point of intersection of the curve C_p with the positive real semiaxis. Thus, For $0 < \omega < 1 - 1/\sqrt{\eta}$, it is obvious that no such region exists since then $\eta > 1/(1-\omega)^2$.

For $1 - 1/\sqrt{\eta} \leqslant \omega < 1$, it is checked that $|z|$ takes its minimum at the point corresponding to $\theta = 0$. So, the closest to 0 point we are seeking is that corresponding to $\theta = 0$.

For $1 \leqslant \omega < 1 + 1/\sqrt{\eta}$, it is known from our analysis that for certain ω's the mapping (2.2) is not conformal for $\eta > \hat{\eta}$. The question that arises is: For the given value of η can one find an $\widehat{\omega}$ for which the mapping ceases to be conformal? To answer it, a further study of Eq. (2.4) must be made. Setting $x = 1 - \omega$, (2.4) becomes

$$f(x) := \eta^2 x^3 + (p-1)\eta x^2 + (p-1)\eta x + 1 = 0. \tag{3.1}$$

It is $f(0) = 1 > 0$ and $f(-1/\eta) = (p-2)(1/\eta - 1) < 0$. By studying the sign of the derivative of f w.r.t. x as a function of x it is concluded that there is precisely one zero of f in the interval in question. This implies that there is precisely one value of ω, denoted by $\widehat{\omega}$, in the interval $(1, 1 + 1/\sqrt{\eta})$ for which the mapping ceases to be conformal. Since the mapping is conformal for $\omega = 1$, it will be conformal for all the values of $\omega \in [1, \widehat{\omega}]$.

For $\widehat{\omega} < \omega \leqslant 1 + 1/\sqrt{\eta}$, the mapping is not conformal. The closest to the origin intersection point of interest corresponds to a $\theta \neq 0$ that can be found only computationally. For this, one sets $\operatorname{Im} z(\eta e^{i\theta}) = 0$, solves for θ and then finds the value of θ that gives the smallest positive $\operatorname{Re} z(\eta e^{i\theta})$.

For $1 + 1/\sqrt{\eta} < \omega < 2$, it is $\eta > 1/(1-\omega)^2$ and there exists *no* region of convergence.

The analysis so far gives the boundary curve of the region of interest in the $(\rho(B^p), \omega)$-plane. For $1 - 1/\sqrt{\eta} \leqslant \omega \leqslant \widehat{\omega}$, this boundary curve can be given analytically by putting $\theta = 0$ in (2.2). So, we obtain

$$\beta_1(\omega) = \frac{\left[1 - (1-\omega)^2\eta\right]^p}{(2-\omega)^2 \omega^p \eta \left[1 + (1-\omega)\eta\right]^{p-2}}, \quad \omega \in \left[1 - \frac{1}{\sqrt{\eta}}, \widehat{\omega}\right]. \tag{3.2}$$

For $\widehat{\omega} < \omega < 1 + 1/\sqrt{\eta}$, the boundary curve $\beta_2(\omega)$ can be found only computationally.

The above analysis gives an answer to the second part of question (i) of the Introduction.

To determine optimal values of ω, we must determine those ω's which maximize η for a given $\rho(B)$ or ω's that maximize $\rho(B)$ for a given η. The above analysis can be used to determine the desired optimal values. Specifically, we must determine ω^* such that

$$\beta(\omega^*) = \max_{\omega} \beta(\omega), \omega \in \left(1 - \frac{1}{\sqrt{\eta}}, 1 + \frac{1}{\sqrt{\eta}}\right), \tag{3.3}$$

where

$$\beta(\omega) = \begin{cases} \beta_1(\omega), \; \omega \in \left(1 - \frac{1}{\sqrt{\eta}}, \widehat{\omega}\right], \\ \beta_2(\omega), \; \omega \in \left(\widehat{\omega}, 1 + \frac{1}{\sqrt{\eta}}\right). \end{cases} \tag{3.4}$$

For this we must find the maximum values of the functions $\beta_1(\omega)$ and $\beta_2(\omega)$. In the following, we work with $\beta_1(\omega)$ only and not with $\beta_2(\omega)$ since the latter can be given only computationally.

To study the function $\beta_1(\omega)$ first we differentiate it w.r.t. ω and then use the transformation $x = 1 - \omega$. After some simple algebraic manipulation we obtain

$$\beta_1'(\omega) \sim (p-2)\eta x^3 + (p+2)\eta x^2 + (p+2)x + (p-2) =: f(x;\eta), \tag{3.5}$$

where a relationship of the form $A \sim B$ denotes that $\mathrm{sign}(A) = \mathrm{sign}(B)$.

To find the sign of $f(x;\eta)$ for $x \in [1-\widehat{\omega}, 1/\sqrt{\eta})$ and $\eta \geqslant 1$ we find first the sign of

$$\begin{aligned} f(x;1) &= (p-2)x^3 + (p+2)x^2 + (p+2)x + (p-2) \\ &= (x+1)[(p-2)x^2 + 4x + (p-2)]. \end{aligned} \tag{3.6}$$

$f(x;1)$ has the simple zero -1 for all $p > 4$, the triple zero -1 for $p = 4$, and the zeros $-2-\sqrt{3}$, -1 and $-2+\sqrt{3}$ for $p = 3$. So, $f(x;1) > 0$, $\forall\, x \in [1-\widehat{\omega}, 1/\sqrt{\eta})$ and $p \geqslant 4$. For $p = 3$ the zero $-2+\sqrt{3}$ lies in the interval $[1-\widehat{\omega}, 1/\sqrt{\eta})$ for some values of η. So, $f(x;1)$ may change sign in this interval. Since, $f(x;1)$ behaves differently for $p = 3$ this case will be examined separately in Section 4. From (3.5) and (3.6) we have that

$$f(x;\eta) = f(x;1) + x^2(\eta - 1)[(p-2)x + (p+2)]. \tag{3.7}$$

Since for $p \geqslant 4$, both terms on the right-hand side of (3.7) are positive we have that $f(x;\eta) > 0, \forall\, x \in [1-\widehat{\omega}, 1/\sqrt{\eta})$. This means that $\beta_1(\omega)$ is a strictly increasing function in $[1-\widehat{\omega}, 1/\sqrt{\eta})$. Consequently,

$$\max_{\omega} \beta_1(\omega) = \beta_1(\widehat{\omega}), \quad \omega \in \left[1-\widehat{\omega}, \frac{1}{\sqrt{\eta}}\right). \tag{3.8}$$

Based on the above analysis we state and prove the following theorem.

Theorem 3.2. *Let $p \geqslant 4$, $\sigma(B^p)$ be nonnegative and $\rho(B)$ (<1) be given. Then there exists a unique root $\widehat{\omega} \in (1,2)$ of the equation*

$$\begin{aligned} \rho(B^p) = {} & \frac{\left[2 + (p-1)(2-\omega) - \left[(p-1)^2(2-\omega)^2 + 4(\omega-1)\right]^{1/2}\right]^p}{2(2-\omega)^2\omega^p\left[-(p-1)(2-\omega) + \left[(p-1)^2(2-\omega)^2 + 4(\omega-1)\right]^{1/2}\right]} \\ & \times \frac{(1-\omega)^p}{\left[2(1-\omega) - (p-1)(2-\omega) + \left[(p-1)^2(2-\omega)^2 + 4(\omega-1)\right]^{1/2}\right]^{p-2}} \end{aligned} \tag{3.9}$$

and the SSOR method converges for all $\omega \in (0, \widehat{\omega}]$ with $\widehat{\omega}$ being the optimal value of ω in this interval. The associated optimal spectral radius is $\rho(S_{\widehat{\omega}}) = 1/\widehat{\eta}$, where $\widehat{\eta}$ is the value of η given by Theorem 2.1.

Proof. From the theory developed we have that for a given $\widehat{\eta} > 1$ there exists a unique value of $\widehat{\omega} \in (1,2)$ given from the expression (2.12) of Theorem 2.1. From (3.2), the pair $\widehat{\omega}$ and $\widehat{\eta}$ corresponds to the largest $\rho(B)$. Conversely, for a given $\rho(B) < 1$ there exists a unique value $\widehat{\omega} \in (1,2)$ and a corresponding value for $\widehat{\eta}$ such that the convergence will be optimal for all $\omega \in (0, \widehat{\omega}]$. (3.9) is obtained using the expressions for $\widehat{\omega}$ and $\widehat{\eta}$ in (3.2). □

3.2. Nonpositive case

An analysis analogous to the one in Section 3.1, but much more complicated this time, can also be done. Because of its many technical details only the main theoretical result is presented. For more, the interested reader is referred to [13].

Theorem 3.3. *Let $p \geqslant 4$, $\sigma(B^p)$ be nonpositive, $\rho(B)$ be given and*

$$\rho_i = \frac{1+(1-\omega_i)^2}{(2-\omega_i)^{2/p}\omega_i^{2-2/p}}, \quad i=1,2 \tag{3.10}$$

with

$$\omega_1 = \frac{2(p-2)^{1/2}}{(p+2)^{1/2}+(p-2)^{1/2}}, \qquad \omega_2 = \frac{2(p-1)^{1/2}}{(p-1)^{1/2}+1}. \tag{3.11}$$

(a) *For any $\rho(B) < \rho_1$, there exists a unique root $\tilde{\omega} \in (0,1)$ of the equation*

$$\rho(B^p) = \frac{\left[2+(p-1)(2-\omega)-\left[(p-1)^2(2-\omega)^2+4(\omega-1)\right]^{1/2}\right]^p}{2(2-\omega)^2\omega^p\left[(p-1)(2-\omega)-\left[(p-1)^2(2-\omega)^2+4(\omega-1)\right]^{1/2}\right]} \times \frac{(1-\omega)^p}{\left[2(1-\omega)-(p-1)(2-\omega)+\left[(p-1)^2(2-\omega)^2+4(\omega-1)\right]^{1/2}\right]^{p-2}} \tag{3.12}$$

and also for any $\rho(B) < \infty$, unless $p \geqslant 15$ and $1 \leqslant \rho(B)$, there exists a unique value $\hat{\hat{\omega}}$ which is the smallest $\omega \in (1,2)$ such that there is an intersection point of the curve C_p at $(-\rho(B^p),0)$ for a $\theta \neq \pi$. Let $\eta^* = \tilde{\eta}$ (resp. $\eta^* = \hat{\hat{\eta}}$) denote the corresponding value of η; if both $\tilde{\omega}$ and $\hat{\hat{\omega}}$ exist let $\eta^* = \max\{\tilde{\eta}, \hat{\hat{\eta}}\}$. Then, if $\tilde{\omega}$ (resp. $\hat{\hat{\omega}}$) exists there is always an interval of ω containing $\tilde{\omega}$ (resp. $\hat{\hat{\omega}}$) in which the SSOR method converges with local optimal spectral radius at $\omega^* = \tilde{\omega}$ (resp. $\omega^* = \hat{\hat{\omega}}$), namely $\rho(S_{\omega^*}) = 1/\eta^*$.

(b) *For any $p \geqslant 4$ and any $\rho(B) \leqslant \rho_2$ the SSOR method converges for all ω in an interval containing both $\tilde{\omega}$ and $\hat{\hat{\omega}}$ with optimal spectral radius $\rho(S_{\omega^*}) = 1/\eta^*$.*

Note. One can be more specific about the intervals of convergence containing $\tilde{\omega}$ (resp. $\hat{\hat{\omega}}$) if one uses the theory in [9, 12], the preceding analysis and Theorem 3.3.

4. The special case $p = 3$

From the analysis of Section 2 and specially from Theorem 2.1 and Corollary 2.2, for $p = 3$, we have that

$$\hat{\eta} = \frac{-(2-\omega)+\left[(2-\omega)^2+\omega-1\right]^{1/2}}{(\omega-1)^2} \quad \text{and} \quad \tilde{\eta} = \frac{(2-\omega)-\left[(2-\omega)^2+\omega-1\right]^{1/2}}{(\omega-1)^2}. \tag{4.1}$$

Following the analysis of Section 3 we try to find the boundary curves of the regions of convergence in the nonnegative case while for the nonpositive case the reader is referred to [13].

4.1. Nonnegative case

For $\omega < 1$, the conclusions are the same as those of the general case. So, for $1 \leqslant \omega \leqslant 1 + 1/\eta$ there exists an $\widehat{\omega}$ corresponding to $\widehat{\eta}$ such that for $\omega \leqslant \widehat{\omega}$ the mapping (2.2) is conformal while for $\omega > \widehat{\omega}$ it is not. Consequently, the boundary curve of the convergence region is that given in (3.4), where

$$\beta_1(\omega) = \frac{\left[1-(1-\omega)^2\eta\right]^3}{(2-\omega)^2\omega^2\eta\left[1+(1-\omega)\eta\right]}, \quad \omega \in \left[1-\frac{1}{\sqrt{\eta}}, \widehat{\omega}\right] \tag{4.2}$$

while $\beta_2(\omega)$ can be found only computationally.

From (3.5) we have that

$$\beta_1'(\omega) \sim \eta x^3 + 5\eta x^2 + 5x + 1 =: f_3(x;\eta) \tag{4.3}$$

It is obtained that $f_3(-1/\eta;\eta) > 0$ and $f_3(0;\eta) > 0$. Differentiating $f_3(x;\eta)$ w.r.t. x we have

$$f_3'(x;\eta) = 3\eta x^2 + 10\eta x + 5. \tag{4.4}$$

Since $f_3'(-1/\eta;\eta) < 0$ and $f_3'(0;\eta) > 0$, $f_3'(x;\eta)$ has one zero $\xi = \left(-5+\sqrt{25-15/\eta}\right)/3 \in \left(-1/\eta, 0\right)$. By plugging ξ in (4.3) we can find, with as much accuracy as we want, an η so that $f_3(\xi;\eta) = 0$, which, to eight significant figures, is 1.3615345. So, for $\eta \in [1, 1.3615345)$, $f_3(\xi;\eta) < 0$ and $f_3(x;\eta)$ has two zeros in $\left(-1/\eta, 0\right)$. To find out which one of the two zeros belongs to the interval $(\widehat{x}, 0)$, where $\widehat{x} = 1 - \widehat{\omega}$, we find the value of η such that $\widehat{x}$ is one of the two aforementioned zeros. Applying the Sylvester's resolvent to the equation $f_3'(x;\eta) = 0$ and to equation (2.5) for $p = 3$, we obtain that $\widehat{x}$ can be a common root of these two equations only if $\eta = 1$. This implies that for all $\eta \in [1, 1.3615345)$ the two zeros belong to the interval $(\widehat{x}, 1)$. For η increasing continuously from 1, these two roots remain in the interval $(\widehat{x}, 1)$ until $\eta = 1.3615345$ when they coincide with the double root ξ. From the value $\eta = 1.3615345$ onwards the two roots become a pair of complex conjugate ones.

Since $\beta_1(\omega)$ is an increasing function at $\omega = 1$, it will be increasing in a neighborhood of 1. So, by increasing ω continuously from 1 to $\widehat{\omega}$ we will pass first through a value of ω, let it be denoted by ω', which will correspond to the local maximum of $\beta_1(\omega)$ and then through a value of ω corresponding to the local minimum.

The previous discussion leads to the conclusion that the optimal value of $\omega \in (1 - 1/\sqrt{\eta}, \widehat{\omega}]$, let it be ω^*, will be given by that value of the pair $\{\omega', \widehat{\omega}\}$ that maximizes $\beta_1(\omega)$.

By putting $\widehat{\omega}$ for ω in (4.2) and $\widehat{\eta}$ for η in (4.1) we obtain

$$\begin{aligned}\beta_1(\widehat{\omega}) &= \frac{\left[(3-\widehat{\omega}) - \left[(2-\widehat{\omega})^2 + (\widehat{\omega}-1)\right]^{1/2}\right]^3}{(2-\widehat{\omega})^2\widehat{\omega}^3\left[-(2-\widehat{\omega}) + \left[(2-\widehat{\omega})^2 + (\widehat{\omega}-1)\right]^{1/2}\right]} \times \frac{(1-\widehat{\omega})^3}{\left[-1 + \left[(2-\widehat{\omega})^2 + (\widehat{\omega}-1)\right]^{1/2}\right]} \\ &=: g(\widehat{\omega}).\end{aligned} \tag{4.5}$$

It is readily seen that $g(\omega)$, defined in (4.5), is a continuous function of $\omega \in (1,2)$. By differentiating $g(\omega)$ and studying the sign of its derivative it can be found after a long manipulation that $\mathrm{d}g(\omega)/\mathrm{d}\omega$ is positive in (1,2) with $\lim_{\omega\to 1^+} g(\omega) = 0$ and $\lim_{\omega\to 2^-} g(\omega) = \frac{27}{32}$. Therefore, $g(\omega)$ is a strictly increasing function in [1,2) which implies that for $\rho(B^3) \in (\frac{27}{32}, 1)$ there is *no* $\widehat{\omega}$ such that $\beta_1(\widehat{\omega}) = \rho(B^3)$. This constitutes one of the main differences of the special case $p = 3$ from the general case $p \geqslant 4$.

4.2. Nonpositive case

The analysis in this case is along the lines of that of the corresponding one of the general case in Section 3.2. It is much more complicated than the ones in Sections 3.2 and 4.1 although some of the very complicated expressions and functions studied in Section 3.2 have now a more concrete form because $p = 3$. Since the analysis in this case is full of technical details and, on the other hand, some of the points raised can be answered only computationally we prefer not to give it here. Instead, the interested reader is referred to our Technical Report [13].

5. Numerical examples and concluding remarks

A number of numerical examples run on a computer are illustrated in Table 1 for $p = 4$ and for selected values of η. In each case, based on the theory developed a systematic search was made with $\omega = 0.00001(0.00001)1.99999$ to find the one for which $\rho(B)$ was as large as possible. The ω found appears as ω_{opt} in the table. Then, we worked the other way around. So, for each computationally obtained ω_{opt}, using the value of $\rho(B)$ that was found, we computed the corresponding value of η. The η's we obtained were very close to the ones originally considered. Some minor discrepancies

Table 1
Case $p = 4$

		Nonnegative		Nonpositive	
η	$\rho(S_\omega)$	ω_{opt}	$\rho(B)$	ω_{opt}	$\rho(B)$
1.01	0.990099	1.75757	0.999825	1.99010	7.08862
1.025	0.975610	1.68152	0.999397	1.97562	4.50044
1.05	0.952381	1.61069	0.998455	1.95242	3.20247
1.1	0.909091	1.52737	0.996048	1.90946	2.29492
1.2	0.833333	1.43243	0.990037	1.83559	1.66714
1.3	0.769231	1.37312	0.983163	1.77491	1.39772
1.4	0.714286	1.33046	0.975863	0.798640	1.30969
1.5	0.666667	1.29762	0.968374	0.810338	1.27384
1.7	0.588235	1.24954	0.953348	0.830029	1.21361
2.0	0.5	1.20194	0.931606	0.852860	1.14327
2.5	0.4	1.15401	0.898763	0.879690	1.05839
3.0	0.333333	1.12473	0.870202	0.898198	0.997075
4.0	0.25	1.09054	0.823413	0.922104	0.911813
5.0	0.2	1.07113	0.786582	0.936897	0.853412
7.5	0.133333	1.04634	0.720355	0.957192	0.760582
10.0	0.1	1.03438	0.674892	0.967603	0.702961
15.0	0.066667	1.02268	0.613961	0.978201	0.630867
20.0	0.05	1.01692	0.573289	0.983574	0.585088

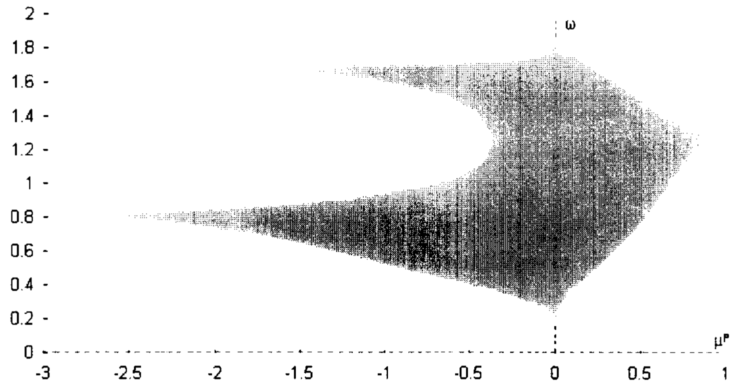

Fig. 2. Convergence region of the general case.

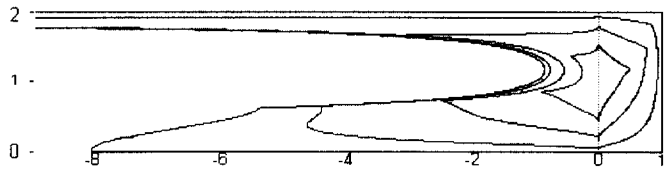

Fig. 3. Special case $p = 3$.

can be attributed to the presence of the round-off errors and the final effect of their propagation during the many complicated computations involved.

Some of the results in the table are depicted in Fig. 2, that is a good representative of the general case, for both the nonnegative and nonpositive cases in the (μ^4, ω)-plane. It shows the region of convergence for $p = 4$ and $\eta = 1.5$. We would like to draw the attention of the reader to the two local maxima in the nonpositive case.

Similarly, Fig. 3 depicts in the (μ^3, ω)-plane the special case $p = 3$ for the values of $\eta = 1,\ 1.1,\ 1.5$ and 3. The outmost curves correspond to $\eta = 1$ and the inmost ones to $\eta = 3$. The reader should note the main difference in the shape of the boundary curve in the nonpositive case for some value of η compared to that in the general case. In particular, as η increases from the value 1 there is a maximum value for some $\omega < 1$ and at the same time it seems that there is a cusp in the boundary curve for another value of $\omega < 1$ greater than the previous one. As η continues on increasing the smaller value of ω corresponding to the maximum tends to the cusp until some value of η where these two ω's coincide. From that point onwards only the value of ω at the cusp appears which corresponds to a local maximum. The appearance of the maximum and the cusp at two different values of ω for some values of η is what distinguishes the behavior of the shape of the boundary in this case for $p = 3$ from the one in the general case for $p \geqslant 4$.

It should be noted that in Figs. 2 and 3 the lower parts of the graphs tend to the point $(0, 0)$ as $\rho(B) \to 0^+$ while their upper parts tend to the point $(0, 2)$. Both these points constitute singular points for all the graphs since the SSOR method does not converge for $\omega = 0$ or 2.

6. Open questions

We conclude this work by giving a brief summary of what was done and by addressing a number of open questions in the form of conjectures.

In the present work we made an effort to determine, among others, the optimal values of the SSOR relaxation factor ω in the p-cyclic consistently ordered case when the pth powers of the eigenvalues of the associated Jacobi iteration matrix were all of the same sign. We think that our effort was successful in the sense that we were able to determine these values analytically when a subinterval of the interval of convergence $(0,2)$ was considered. Due to the very complicated nature of the functions that describe various parts of the boundaries of the convergence region we were not able to extend analytically our theory to cover the whole interval $(0,2)$. However, strong numerical evidence based on numerous experiments and computer graphics for various values of all of the parameters involved (a very characteristic sample of the corresponding situations can be seen in Figs. 2 and 3) have convinced us that the optimal values we found are also the overall optimal ones.

There are a number of open questions one should answer in order to make a complete theoretical proof. These are raised explicitly or implicitly in this work and also, mainly, in [13]. Here we present only the most basic ones in the form of conjectures.

Conjecture 1. *In the nonnegative case, for* $p \geqslant 4$, *and under the assumptions of Theorem* 3.2, *the optimal* ω *is the value of* $\widehat{\omega}$ *given in this theorem.*

Conjecture 2. *In the nonpositive case, for* $p \geqslant 4$, *and under the assumptions of Theorem* 3.3, *the optimal* ω *is the value of* ω^* *given in this theorem.*

Conjecture 3. *In the nonnegative case, for* $p = 3$, *and under the assumptions of the analysis in Section* 4.1, *the optimal* ω *is the value of* ω^* *given in this analysis.*

Conjecture 4. *In the nonpositive case, for* $p = 3$, *and under the assumptions of the analysis presented in Section* 4.2 *of* [13], *the optimal* ω *is the value of* ω^* *out of the triad* $(\widehat{\omega}, \widehat{\widehat{\omega}}, \tilde{\tilde{\omega}})$ *that maximizes the corresponding value for the boundary curve.* (*See* [13] *for the notation used and for further explanations.*)

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