NORTH-HOLLAND

On the Exact p-Cyclic SSOR Convergence Domains

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ABSTRACT

Suppose that $A \in \mathbb{C}^{n,n}$ is a block p-cyclic consistently ordered matrix, and let B and S_ω denote, respectively, the block Jacobi and the block symmetric successive overrelaxation (SSOR) iteration matrices associated with A. Neumaier and Varga found [in the $(\rho(|B|), \omega)$ plane] the exact convergence and divergence domains of the SSOR method for the class of H-matrices. Hadjidimos and Neumann applied Rouché's theorem to the functional equation connecting the eigenvalue spectra $\sigma(B)$ and $\sigma(S_\omega)$ obtained by Varga, Niethammer, and Cai, and derived in the $(\rho(B), \omega)$ plane the convergence domains for the SSOR method associated with p-cyclic consistently ordered matrices, for any $p \geqslant 3$. In the present work it is further assumed that the eigenvalues of B^p are real of the same sign. Under this assumption the *exact* convergence domains in the $(\rho(B), \omega)$ plane are derived in both the nonnegative and the nonpositive cases for any $p \geqslant 3$.

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1. INTRODUCTION

Consider the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n,n}$ and $x, b \in \mathbb{C}^n$, and suppose that A is written in the $p \times p$ block form

$$A = D(I - L - U) \tag{1.2}$$

with D being a $p \times p$ block diagonal invertible matrix and L and U being strictly lower and strictly upper triangular matrices, respectively. Suppose also that for the solution of (1.1)–(1.2) the symmetric successive overrelaxation (SSOR) iterative method (see, e.g., [14, 16, 1]) is used. The SSOR method is defined by

$$x^{(m+1/2)} = (I - \omega L)^{-1}[(1-\omega)I + \omega U]x^{(m)} + \omega(I - \omega L)^{-1}b,$$
$$x^{(m+1)} = (I - \omega U)^{-1}[(1-\omega)I + \omega L]x^{(m+1/2)} + \omega(I - \omega U)^{-1}b,$$
$$m = 1, 2, \ldots, \tag{1.3}$$

where $x^{(0)} \in \mathbb{C}^n$ is arbitrary and $\omega \in (0, 2)$ is the relaxation factor. The block SSOR iteration matrix associated with A, relative to its block partitioning, is given by

$$S_\omega := (I - \omega U)^{-1}[(I - \omega)I + \omega L](I - \omega L)^{-1}[(I - \omega)I + \omega U]. \tag{1.4}$$

Let $B := L + U$ be the block Jacobi matrix associated with A. If A is block p-cyclic consistently ordered, then without loss of generality B may be assumed to have the block form

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & B_1 \\ B_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & B_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_{p-1} & 0 \end{bmatrix}. \tag{1.5}$$

It is well known that the sets of eigenvalues μ of B (or of B^T) and λ of S_ω satisfy the functional equation obtained by Varga, Niethammer, and Cai [15],

$$\left[\lambda - (1-\omega)^2\right]^p = \lambda(\lambda + 1 - \omega)^{p-2}(2-\omega)^2 \omega^p \mu^p. \tag{1.6}$$

It is noted that (1.6) generalized the corresponding relationship for $p = 2$ (see [4, 11]) and was later generalized in [3] to cover the entire class of p-cyclic, not necessarily consistently ordered matrices.

Recently, Hadjidimos and Neumann [7] have found in the (ν, ω) plane, with $\nu = \rho(B)$ and $\rho(\cdot)$ denoting spectral radius, the domain of convergence for the SSOR method for block p-cyclic consistently ordered matrices A, $p \geqslant 3$. Later the same authors generalized their research to the entire class of p-cyclic matrices [8]. In the analyses in [7, 8] the application of Rouché's theorem (see, e.g., [10, 13]) led to the determination of the convergence domains. The main result of [7] is given in Theorem 1.1, and a typical SSOR convergence domain is depicted in Figure 1.

THEOREM 1.1. *Let A be a nonsingular block p-cyclic consistently ordered matrix, $p \geqslant 3$. Let B and S_ω be the block Jacobi and the block SSOR iteration matrices associated with A and given in* (1.5) *and* (1.4) *respectively. Suppose the $\rho(B) = \nu$. Then $\rho(S_\omega) < 1$ provided that $(\nu, \omega) \in R(p)$, where $R(p)$ is the region in the (ν, ω) plane defined by*

$$R(p) := \begin{cases} 0 < \omega \leqslant 1, 0 \leqslant \nu < 1 =: \nu_1(\omega), \\ 1 \leqslant \omega \leqslant \hat{\omega}, 0 \leqslant \nu < \dfrac{1 + (1-\omega)^2}{(2-\omega)^{2/p}\omega^{2-2/p}} =: \nu_2(\omega), \\ \hat{\omega} \leqslant \omega < 2, 0 \leqslant \nu \\ \qquad < \dfrac{\left[1 + (1-\omega)^4 - 2(1-\omega)^2\varphi\right]^{1/2}}{\omega(2-\omega)^{2/p}\left[1 + (1-\omega)^2 + 2(1-\omega)\varphi\right]^{1/2-1/p}} \\ =: \nu_3(\omega) \end{cases} \tag{1.7}$$

where

$$\hat{\omega} := \frac{2(-\hat{y}+2)^{1/2}}{(-\hat{y}+2)^{1/2} + (-\hat{y}-2)^{1/2}}, \qquad \hat{y} = -\frac{p + (9p^2 - 16p)^{1/2}}{2(p-2)}, \tag{1.8}$$

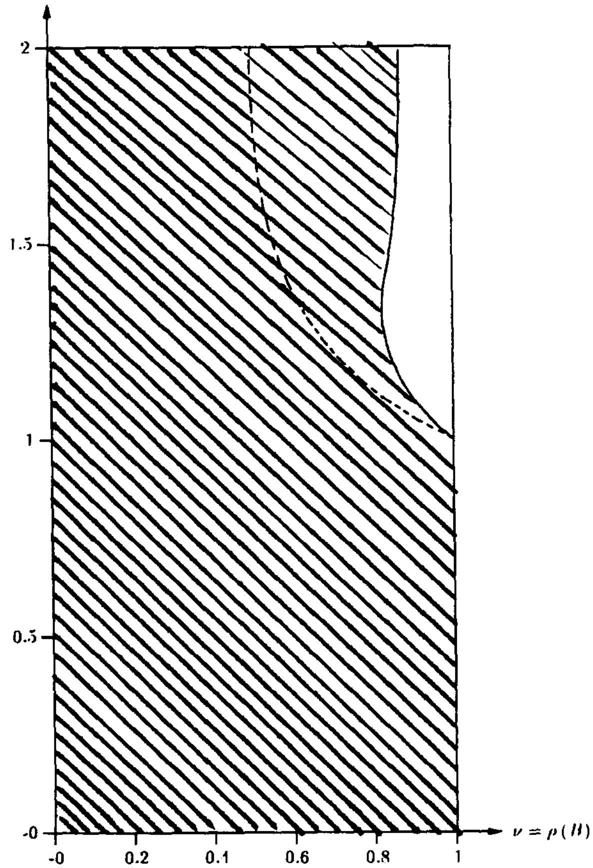

FIG. 1. Convergence domain of SSOR for p-cyclic matrices ($p = 5$).

$$\varphi := \varphi(\omega) := \tfrac{1}{4}\left[-(p-2)y^2 - py + 2(p-2)\right],$$
$$y := y(\omega) = 1 - \omega + \frac{1}{1-\omega}. \tag{1.9}$$

NOTE. It is worth pointing out that on the right boundary of $R(p)$ given by the union of the three arcs $\nu_1(\omega)$, $\nu_2(\omega)$, and $\nu_3(\omega)$ of (1.7) the following hold:

(i) When $|\mu| = 1 \equiv \nu_1(\omega)$, a necessary and sufficient condition for $\lambda \in \sigma(S_\omega)$, $|\lambda| = 1$, is that $\lambda = 1$ and $\mu^p = 1$. This property can be extended to all $\omega \in (0, 2)$.

(ii) When $|\mu| = \nu_2(\omega)$, a necessary and sufficient condition for $\lambda \in \sigma(S_\omega)$, $|\lambda| = 1$, is that $\lambda = -1$ and $\mu^p = -[1 + (1-\omega)^2]^p/(2-\omega)^2\omega^{2p-2}$, a property that can also be extended to cover all $\omega \in (0, 2)$.

It is noted that as $p \to \infty$, then, from (1.8), $\hat{y} \to -2^-$, $\hat{\omega} \to 2^-$, and the right boundary of $R(p)$ in (1.7) tends to $\nu(\omega) = [1 + (1-\omega)^2]/\omega^2$ (or $\omega = 2/[1 + (2\nu - 1)^{1/2}]$, $\frac{1}{2} < \nu \leqslant 1$) (see dashed line in Figure 1). In this limiting case $R(p)$ describes the point SSOR convergence domain for the entire class of H-matrices A found by Neumaier and Varga [12]. An open question in [12] regarding convergence on the upper part of the right boundary of the region was settled in [6]. We also note here that ν in [12] and [6] denotes $\nu = \rho(|B|)$ and *not* $\nu = \rho(B)$.

In this manuscript we obtain, in the (ν, ω) plane, the *exact* SSOR convergence domains for (block) p-cyclic consistently ordered matrices for which $\sigma(B^p)$ is (i) nonnegative and (ii) nonpositive, with $\sigma(\cdot)$ denoting the spectrum. However, by Theorem 1.1 and its note, we notice that we actually seek the following:

(i) In the nonnegative case, the right boundary of the domain in question for $1 < \omega < 2$. Obviously, this boundary must lie strictly to the right of $\nu(\omega) = [1 + (1-\omega)^2]/\omega^2$ and to the left of $\nu_1(\omega) = 1$.

(ii) In the nonpositive case, the corresponding right boundary for $0 < \omega < 1$ and $\hat{\omega} < \omega < 2$. This boundary must lie strictly to the right of $\nu_1(\omega) = 1$ and to the left of $\nu_2(\omega)$, for $0 < \omega < 1$, while for $\hat{\omega} < \omega < 2$ it must be strictly to the right of $\nu(\omega) = [1 + (1-\omega)^2]/\omega^2$ and to the left of $\nu_2(\omega)$.

To derive the parts of the desired right boundaries, our study will have as a starting point the functional equation (1.6), which, except for some trivial cases, can be rewritten as

$$\mu^p = \frac{\left[\lambda - (1-\omega)^2\right]^p}{(2-\omega)^2\omega^p\lambda(\lambda + 1 - \omega)^{p-2}}. \tag{1.10}$$

The basic idea is to use (1.10) and find, for either nonnegative or nonpositive spectra $\sigma(B^p)$, all possible pairs (μ^p, ω) [or equivalently (ν, ω), with $\nu = |\mu|$], where μ^p belongs to a real interval having as one of its endpoints the point 0, such that $|\lambda| < 1$. For this we set

$$|\lambda| = 1 \quad \Leftrightarrow \quad \lambda = e^{i\theta}, \quad \theta \in [0, \pi], \tag{1.11}$$

and replace λ in (1.10) by the expression in (1.11) to obtain

$$F := F(\omega, \theta) := \frac{\left[e^{i\theta} - (1-\omega)^2\right]^p}{(2-\omega)^2 \omega^p e^{i\theta}(e^{i\theta} + 1 - \omega)^{p-2}}. \tag{1.12}$$

In Section 2, after we identify our problem, a complete study of the function F for each fixed $\omega \in (0,2)$ and for all $\theta \in [0, \pi]$ is made. In Sections 3 and 4 the application of the results obtained in Section 2 allows us to determine the *exact* domains of convergence of the SSOR method in the nonnegative and the nonpositive case, respectively. Finally, in Section 5 some remarks are made, and some particular cases treated in the previous sections are further investigated.

2. STUDY OF THE FUNCTION F IN (1.12)

2.1. *Introduction*

Before we begin with the study of the function $F(\omega, \theta)$, we shall identify our problem.

Consider the two transformations below, which are inverse of each other:

$$y := y(\omega) := 1 - \omega + \frac{1}{1-\omega}, \qquad \omega \in (0,2) \setminus \{1\}, \tag{2.1}$$

$$\omega := \omega(y) := \begin{cases} \dfrac{2 - y + \sqrt{y^2 - 4}}{2}, & y \in (2, +\infty), \\ \dfrac{2 - y - \sqrt{y^2 - 4}}{2}, & y \in (-\infty, -2). \end{cases} \tag{2.2}$$

REMARK. The function F and those to be defined are given in terms of ω because we are interested in domains in the (ν, ω) plane. However, use of $y = y(\omega)$ greatly simplifies our analysis.

The function $F(\omega, \theta)$ can be written explicitly as

$$F(\omega, \theta) = \operatorname{Re} F + i \operatorname{Im} F. \tag{2.3}$$

Furthermore,

$$\operatorname{Re} F(\omega, 0) = 1 > 0, \tag{2.4}$$

$$\operatorname{Re} F(\omega, \pi) = -\frac{y^p}{(y+2)(y-2)^{p-1}} < 0. \tag{2.5}$$

Also all other values of $\theta \in (0, \pi)$, if any, such that $\operatorname{Im} F = 0$ have to be found.

Let θ^+ be the set of all $\theta \in [0, \pi)$ such that

$$\operatorname{Im} F(\omega, \theta) = 0, \qquad \operatorname{Re} F(\omega, \theta) \geqslant 0. \tag{2.6}$$

Let also θ^- be the set of all $\theta \in (0, \pi]$ such that

$$\operatorname{Im} F(\omega, \theta) = 0, \qquad \operatorname{Re} F(\omega, \theta) \leqslant 0. \tag{2.7}$$

Then our problem is twofold. Specifically, for the nonnegative case, determine $\theta \in \theta^+$ such that

$$\operatorname{Re} F(\omega, \theta) \text{ is a minimum}, \tag{2.8}$$

and for the nonpositive case, determine $\theta \in \theta^-$ such that

$$\operatorname{Re} F(\omega, \theta) \text{ is a maximum}. \tag{2.9}$$

In the subsequent analysis and for each fixed $\omega \in (0, 2) \setminus \{1\}$ we find all p such that besides the obvious solution $\theta = 0$ for the problem (2.6) [$\theta = \pi$ for (2.7)], there exists at least one more $(0 \neq)\, \theta \in \theta^+$ [$(\pi \neq)\, \theta \in \theta^-$] that solves the problem (2.8) [(2.9)].

2.2. *Study of $F(\omega, \theta)$*

Our analysis is greatly facilitated if we rewrite the function $F(\omega, \theta)$ in (1.12) in the form below:

$$F = F_1 F_2^{p-2}, \tag{2.10}$$

where

$$F_1 := F_1(\omega, \theta) := \frac{\left[e^{i\theta} - (1-\omega)^2\right]^2}{(2-\omega)^2 \omega^2 e^{i\theta}}, \tag{2.11}$$

$$F_2 := F_2(\omega, \theta) := \frac{e^{i\theta} - (1-\omega)^2}{\omega(e^{i\theta} + 1 - \omega)}. \tag{2.12}$$

Then we introduce the functions

$$a_1 := a_1(\omega, \theta) := \arg F_1, \qquad a_2 := a_2(\omega, \theta) := \arg F_2,$$

$$a := a(\omega, \theta) := \arg F = a_1 + (p-2)a_2, \tag{2.13}$$

$$r_1 := r_1(\omega, \theta) := |F_1|, \quad r_2 := r_2(\omega, \theta) := |F_2|, \quad r := r(\omega, \theta) = r_1 r_2^{p-2},$$

and distinguish the two cases $\omega \in (0, 1)$ and $\omega \in (1.2)$.

2.2.1 *Case* $\omega \in (0, 1)$. From the expressions (2.10)–(2.13) and in view of (2.1), it can be readily obtained that

$$\sin a_1 = \frac{y(y^2-4)^{1/2}\sin\theta}{y^2-2-2\cos\theta}, \qquad \cos a_1 = \frac{(y^2-2)\cos\theta - 2}{y^2-2-2\cos\theta}, \tag{2.14}$$

$$\sin a_2 = \frac{(y+2)^{1/2}\sin\theta}{(y^2-2-2\cos\theta)^{1/2}(y+2\cos\theta)^{1/2}}, \tag{2.15a}$$

$$\cos a_2 = \frac{(y-2)^{1/2}(y+1+\cos\theta)}{(y^2-2-2\cos\theta)^{1/2}(y+2\cos\theta)^{1/2}}, \tag{2.15b}$$

$$r_1 = \frac{y^2-2-2\cos\theta}{(y+2)(y-2)}, \qquad r_2 = \left(\frac{y^2-2-2\cos\theta}{(y-2)(y+2\cos\theta)}\right)^{1/2}, \tag{2.16}$$

$$r = \frac{(y^2-2-2\cos\theta)^{p/2}}{(y-2)^{p/2}(y+2)(y+2\cos\theta)^{p/2-1}}. \tag{2.17}$$

Below, two important theorems are proved, where to simplify some relationships we shall use the new relation $A \sim B$ to denote that the expressions A and B are of the same sign.

THEOREM 2.1. *For a fixed* $\omega \in (0, 1)$, a *of* (2.13) *strictly increases with* $\theta \in [0, \pi]$ *if* $\omega \in (\omega^{**}, 1)$. *On the other hand, if* $\omega \in (0, \omega^{**})$, *then* a *strictly increases with* $\theta \in [0, \theta_0]$ *and strictly decreases with* $\theta \in [\theta_0, \pi]$. *Moreover,* $a(\omega, 0) = 0$, $a(\omega, \pi) = \pi$, *while*

$$\omega^{**} = \frac{2(p-2)^{1/2}}{(p+2)^{1/2} + (p-2)^{1/2}} \tag{2.18}$$

and

$$\theta_0 = \arccos\left(-\frac{y^2 + p - 2}{py + p - 2} \right) \in (0, \pi). \tag{2.19}$$

Proof. Differentiating a of (2.13) w.r.t. $\theta \in [0, \pi]$, we obtain

$$\frac{\partial a}{\partial \theta} = \frac{(y^2 - 4)^{1/2}[(py + p - 2)\cos\theta + y^2 + p - 2]}{(y^2 - 2 - 2\cos\theta)(y + 2\cos\theta)}. \tag{2.20}$$

Obviously,

$$\frac{\partial a}{\partial \theta} \sim (py + p - 2)\cos\theta + y^2 + p - 2, \tag{2.21}$$

which gives

$$\left.\frac{\partial a}{\partial \theta}\right|_{\theta=0} \sim y^2 + py + 2(p - 2) > 0, \qquad \left.\frac{\partial a}{\partial \theta}\right|_{\theta=\pi} \sim y(y - p). \tag{2.22}$$

From (2.20)–(2.22), for $y > y^{**} = p$, $\partial a/\partial\theta$ cannot vanish in $(0, \pi]$, while for $y \leqslant y^{**}$, $\partial a/\partial\theta$ does vanish for $\theta = \theta_0$ given by (2.19). From (2.2) it is found that $y^{**} = p$ corresponds to ω^{**} given by (2.18). Considering the variation of the sign of $\partial a/\partial\theta$, the assertions of the present theorem are readily verified. ■

THEOREM 2.2. *For a fixed* $\omega \in (0, 1)$, r *in* (2.13) *strictly increases with* $\theta \in [0, \pi]$. *Moreover,*

$$r(\omega, 0) = 1, \qquad r(\omega, \pi) = \frac{y^p}{(y + 2)(y - 2)^{p-1}}. \tag{2.23}$$

Proof. The proof is easy (See Theorem 2.4 of [9]). ■

2.2.2. *Case* $\omega \in (1,2)$. This time, in view of (2.1), $y \in (-\infty, -2)$. Working in exactly the same way as in Section 2.2.1, we obtain almost identical expressions to those in (2.14)–(2.17), which are given below:

$$\sin a_1 = -\frac{y(y^2-4)^{1/2}\sin\theta}{y^2-2-2\cos\theta}, \qquad \cos a_1 = \frac{(y^2-2)\cos\theta - 2}{y^2-2-2\cos\theta}, \tag{2.24}$$

$$\sin a_2 = -\frac{(-y-2)^{1/2}\sin\theta}{(y^2-2-2\cos\theta)^{1/2}(-y-2\cos\theta)^{1/2}}, \tag{2.25a}$$

$$\cos a_2 = -\frac{(-y+2)^{1/2}(y+1+\cos\theta)}{(y^2-2-2\cos\theta)^{1/2}(-y-2\cos\theta)^{1/2}}, \tag{2.25b}$$

$$r_1 = \frac{y^2-2-2\cos\theta}{(y+2)(y-2)}, \tag{2.26a}$$

$$r_2 = \left(\frac{y^2-2-2\cos\theta}{(y+2)(y-2)}\right)^{1/2} \tag{2.26b}$$

$$r = \frac{(y^2-2-2\cos\theta)^{p/2}}{(-y+2)^{p/2}(-y-2)(-y-2\cos\theta)^{p/2-1}}. \tag{2.26c}$$

Again, statements corresponding to those in Section 2.2.1 can be proved. More specifically:

THEOREM 2.3. *Suppose $\omega \in (1,2)$ is fixed. Then for $p = 3, 4$ the function a in* (2.13) *strictly increases with $\theta \in [0, \pi]$. For $p \geqslant 5$, a strictly increases with $\theta \in [0, \pi]$ for any $\omega \in (1, \omega^*]$, while if $\omega \in [\omega^*, 2)$, then a strictly decreases for $\theta \in [0, \theta_0]$ and strictly increases for $\theta \in [\theta_0, \pi]$. One has $a(\omega, 0) = 0$ and $a(\omega, \pi) = \pi$. The value of θ_0 is given again by* (2.19), *while*

$$\omega^* = \frac{2p^{1/2}}{p^{1/2}+(p-4)^{1/2}}. \tag{2.27}$$

Proof. We work in an analogous way to the proof of Theorem 2.1. Thus, (2.21) and (2.22) are obtained. Because $y < -2$, the first expression in (2.22) changes sign at $y^* = -(p-2)$ provided $p \geqslant 5$. For $p = 3, 4$, the first expression in (2.22) is positive, implying that the function a strictly increases with $\theta \in [0, \pi]$. For $p \geqslant 5$, we have $\partial a/\partial\theta|_{\theta=0} > 0$ for $y < y^*$. Hence, a strictly increases with $\theta \in [0, \pi]$. Since $\partial a/\partial\theta|_{\theta=0} < 0$ for $y > y^*$, $\partial a/\partial\theta = 0$ has a unique root θ_0 given by (2.19). Obviously, the monotonicity of the function a in the two subintervals of ω directly follows. Also ω^* in (2.27) is obtained from (2.2) for $y = y^*$. ■

THEOREM 2.4. *Suppose $\omega \in (1,2)$ if fixed. Then r in (2.13) strictly decreases for $\theta \in [0, \pi]$ if $\omega \in (1, \hat{\omega}]$. If $\omega \in [\hat{\omega}, 2)$, then r strictly decreases for $\theta \in [0, \theta_1]$ and strictly increases for $\theta \in [\theta_1, \pi]$. Here $\hat{\omega}$ is given by*

$$\hat{\omega} = \frac{2(-\hat{y}+2)^{1/2}}{(-\hat{y}+2)^{1/2} + (-\hat{y}-2)^{1/2}}, \qquad \hat{y} = -\frac{p + (9p^2 - 16p)^{1/2}}{2(p-2)}, \tag{2.28}$$

and θ_1 by

$$\theta_1 = \arccos\left(-\frac{(p-2)y^2 + py - 2(p-2)}{4}\right). \tag{2.29}$$

Moreover $\hat{y} > y^*$.

NOTE. The values in (2.28) are the ones in (1.8), obtained in [7].

Proof. Differentiating r in (2.26), we obtain

$$\frac{\partial r}{\partial \theta} \sim -4\cos\theta - (p-2)(y^2-2) - py. \tag{2.30}$$

From (2.30), $\partial r/\partial\theta > 0$ if and only if $\theta \in (\theta_1, \pi)$, with θ_1 given by (2.29). Since

$$\lim_{y \to -2^-}\left(-\frac{(p-2)y^2 + py - 2(p-2)}{4}\right) = 1,$$

the existence of a unique $\theta_1 \in (0, \pi)$ is guaranteed if and only if

$$-\frac{(p-2)y^2 + py - 2(p-2)}{4} > -1,$$

which, in turn, holds if and only if $y > \hat{y}$, where $\hat{y}$ is given by (2.28). The monotonicity of r in the intervals stated are consequences of the sign of $\partial r/\partial\theta$ in (2.30). Finally, it can be checked that $\hat{y} > y^*$. ∎

3. THE NONNEGATIVE CASE

From the analysis in Sections 1 and 2.1, to derive the right boundary of the convergence domain one has to solve the problem (2.6), (2.8) for any fixed $\omega \in (0, 2)$ (and any fixed $p \geqslant 3$). From [7], for $\omega \in (0, 1]$, $\theta = 0$ is the only element of θ^+. So the corresponding right boundary is given by

$$\nu_1(\omega) = 1, \qquad \omega \in (0, 1]. \tag{3.1}$$

We concentrate then on $\omega \in (1, 2)$.

From Theorem 2.3, for $p = 3, 4$, $\theta = 0$ is the only $\theta \in \theta^+$ satisfying (2.8). Hence the right boundary in (3.1) is also the right boundary of the convergence domain for all $\omega \in (1, 2)$, and the convergence domain $R^+(p)$, $p = 3, 4$, is the whole rectangle with vertices $(0, 0)$, $(1, 0)$, $(1, 2)$, and $(2, 0)$, except its bottom, right, and top sides. (Note: The result for $p = 3$ was known [5, 2].)

For $p \geqslant 5$, from Theorem 2.3 we have that for a fixed $\omega \in (1, \omega^*)$ the only solution to (2.6), (2.8) is $\theta = 0$. So the arc of the right boundary is given by (3.1). Also, we have that for a fixed $\omega \in (\omega^*, 2)$, $a(\omega, \theta)$ strictly decreases in $[0, \theta_0]$ and strictly increases in $[\theta_0, \pi]$, with $a(\omega, 0) = 0$, $a(\omega, \pi) = \pi$. This implies that there is at least one value of $\theta \in (\theta_0, \pi)$ such that $\theta \in \theta^+ \backslash \{0\}$. The question that arises is the following. Among all $\theta \in \theta^+ \backslash \{0\}$ is there one that satisfies (2.8)?

For $\omega \in (\omega^*, \hat{\omega}]$ the answer can be given immediately by Theorem 2.4, because $r = |F(\omega, \theta)|$ strictly decreases for $\theta \in [0, \pi]$. Therefore among all $\theta \in \theta^+ \backslash \{0\}$ there will be one that will satisfy (2.8).

To proceed in the case of $\omega \in (\hat{\omega}, 2)$ we prove four lemmas which are useful in the sequel.

LEMMA 3.1. *There exists a value of $y = \bar{y} \in (\hat{y}, -2)$ such that for all $y \in (\bar{y}, -2)$ there exists a $\theta_2 \in (\theta_1, \pi)$ satisfying*

$$\cos\theta_2 = \cos\theta_1 - \frac{(p-2)(y^2 + y - 2)}{4}$$
$$= \frac{-(p-2)y^2 - (p-1)y + 2(p-2)}{2}. \tag{3.2}$$

Proof. From (3.2), $\cos\theta_2$ strictly increases with $y \in (\hat{y}, -2)$. Since $\cos\theta_2|_{y=-2} = 1$, θ_2 exists if and only if the rightmost expression in (3.2) is greater than -1, or if and only if

$$y > \bar{y} := \frac{-(p-1) - (9p^2 - 28p + 17)^{1/2}}{2(p-2)}. \tag{3.3}$$

It can be readily checked that $\bar{y} \in (\hat{y}, 2)$ and that $\theta_2 \in (\theta_1, \pi)$, which completes the proof. ∎

LEMMA 3.2. *For* $5 \leqslant p \leqslant 24$, *one has* $a(\omega, \theta_2) > 0$ *for all* $y \in (\bar{y}, -2)$.

Proof. By using (3.2) in (2.24), (2.25) we obtain

$$\cos a_1|_{\theta=\theta_2} = \frac{(2-y)^{1/2}[(p-2)y - (p-1)]}{2(p-1)^{1/2}(p-2)^{1/2}(y-1)} \tag{3.4}$$

and

$$\cos a_2|_{\theta=\theta_2} = \frac{-(p-2)y^3 + (p-3)y^2 + 2(p-1)y - 2(p-1)}{2(p-1)(y-1)}, \tag{3.5}$$

respectively. Differentiating (3.4), (3.5) w.r.t. y, we have

$$\frac{\partial}{\partial y}(\cos a_1|_{\theta=\theta_2}) \sim y\left[-2(p-2)y^2 + (4p-9)y - 2(p-3)\right] > 0 \tag{3.6}$$

and

$$\frac{\partial}{\partial y}(\cos a_2|_{\theta=\theta_2}) \sim -(p-2)y^2 + (2p-5)y - (p-5) < 0, \tag{3.7}$$

with the inequalities holding for all $p \geqslant 5$. The inequalities (3.6), (3.7) together with $\cos a_1|_{\theta=\theta_2} > 0$ and $\cos a_2|_{\theta=\theta_2} < 0$ imply that both $a_1(\omega, \theta_2)$

and $a_2(\omega, \theta_2)$ are strictly decreasing functions of y. So is $a(\omega, \theta_2)$. It can be checked that for $p \geqslant 5$ the largest value of p giving the smallest positive value of $a(\omega, \theta_2)$, and corresponding to $y = -2$, which is $a(2, \theta_2) \approx 0.0206$, is $p = 24$. ∎

LEMMA 3.3. *The function $r(\omega, \theta_2)$ is given by*

$$r(\omega, \theta_2) = -\frac{(p-1)^{p/2}(y-1)}{(p-2)^{p/2-1}(2-y)^{p/2}} \tag{3.8}$$

and is a strictly increasing function of $y \in (\bar{y}, -2)$.

Proof. The proof is easy (see Lemma 3.3 of [9]). ∎

LEMMA 3.4. *The function $F(\omega, \pi)$ is given by*

$$F(\omega, \pi) = -\frac{(-y)^{p}}{(2-y)^{p-1}(-y-2)} \tag{3.9}$$

and strictly decreases for all $y \in [\hat{y}, -2)$ with $\lim_{y \to -2^-} F(\omega, \pi) = -\infty$.

Proof. The proof is easy (see Lemma 3.4 of [9]). ∎

From Theorem 2.4, for a fixed $\omega \in (\hat{\omega}, 2)$, $r(\omega, \theta)$ strictly decreases for θ in $[0, \theta_1]$ and strictly increases in $[\theta_1, \pi]$. Its maximum value is then attained at either 0 or π. So, if $r(\omega, \pi) < 1\,[= r(\omega, 0)]$, then $r(\omega, \theta) < 1$, $\theta \in (0, \pi]$. Since, by Lemma 3.4, $r(\omega, \pi)\ [= -F(\omega, \pi)]$ strictly increases with $y \in (\hat{y}, -2]$, then $r(\omega, \pi) < 1$ for all $y \in (\hat{y}, \bar{y}]$, if $r(\bar{\omega}, \pi) < 1$, where $\bar{\omega}$ is the value of $\omega \in (1, 2)$ that gives $\bar{y}$. As can be checked, $r(\bar{\omega}, \pi) < 1$ for all $5 \leqslant p \leqslant 24$. This implies that there is a value of $\theta \in \theta^{+}\backslash\{0\}$ that satisfies (2.6) and (2.8) for all $y \in (\hat{y}, \bar{y}]$.

For $y \in (\bar{y}, -2)$, from Lemma 3.2, the real positive value of $F(\omega, \theta)$ corresponds to a $\theta \in (0, \theta_2]$. Thus if $r(\omega, \theta_2) < 1$ then $r(\omega, \theta) < 1$ for all $\theta \in (0, \theta_2]$. Since, from Lemma 3.3, $r(\omega, \theta_2)$ increases w.r.t. y, then $r(2, \theta_2) < 1$ will imply $r(\omega, \theta_2) < 1$ for all $y \in (\bar{y}, -2)$. By direct computation, it can be verified that the values $r(\bar{\omega}, \pi)$ and $r(2, \theta_2)$ are indeed less than 1 for all $5 \leqslant p \leqslant 24$.

The analysis so far effectively shows that for any $5 \leqslant p \leqslant 24$ and for each $\omega \in (\omega^*, 2)$ there exists a value of $\theta \in \theta^+ \backslash \{0\}$ that satisfies (2.8). For this value of θ, $F(\omega, \theta) < 1$. Consequently, the right boundary of the convergence domain will be given by an expression of the form

$$\nu_1' = [F(\omega, \theta)]^{1/p}, \qquad \omega \in (\omega^*, 2). \tag{3.10}$$

A typical convergence domain for $5 \leqslant p \leqslant 24$ is illustrated in Figure 2.

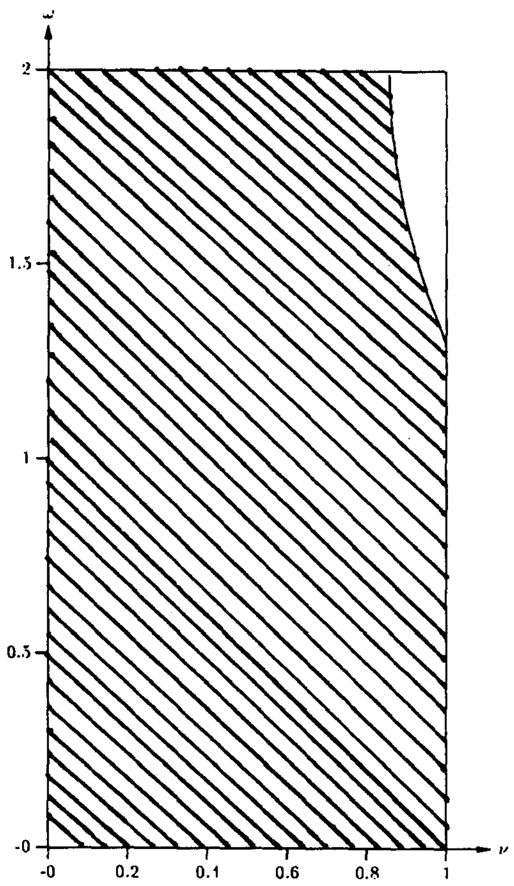

FIG. 2. Nonnegative case ($p \geqslant 5$).

For $p \geqslant 25$ we study the sequences of $a_1(\hat{\omega}, \hat{\theta}_0)$, $a_2(\hat{\omega}, \hat{\theta}_0)$, and $a(\hat{\omega}, \hat{\theta}_0)$, where $\hat{\omega}$ and $\hat{\theta}_0$ are given by (2.28) and (2.19) with $y = \hat{y}$, as functions of p. It can be found that $a(\hat{\omega}, \hat{\theta}_0)|_{p=25} \approx -7.4578 < -2\pi$. This means that there are more than one real nonnegative value of $F(\omega, \theta)$ for $\theta \in (0, \pi)$, with at least one of them less than 1. This is because $r(\omega, \theta)$ strictly decreases in $(0, \theta_1)$ and $\theta_1 > \theta_0$. So for $26 \leqslant p \leqslant 30$ we have exactly the same conclusion as before, since $a(\hat{\omega}, \hat{\theta}_0)$ strictly decreases as a function of p. For $p = 31$, we can find that $(p-2)a_2(\hat{\omega}, \hat{\theta}_0)|_{p=31} \approx -9.5058 < -3\pi$, and since $0 < a_1(\hat{\omega}, \hat{\theta}_0)|_{p=31} < \pi$, we have $a(\hat{\omega}, \hat{\theta}_0)|_{p=31} = a_1(\hat{\omega}, \hat{\theta}_0)|_{p=31} + (p-2)a_2(\hat{\omega}, \hat{\theta}_0)|_{p=31} < -2\pi$. Therefore we reach the same conclusion. For any $p > 31$, we note that $a_2(\hat{\omega}, \hat{\theta}_0)$ strictly decreases, so the same conclusion follows. Thus, the right boundary for $\omega \in (\omega^*, 2)$ is given by (3.10).

We summarize the analysis in this section in the following statement.

THEOREM 3.5. *For $p = 3, 4$, the right boundary of the convergence domain $R^+(p)$ is given by*

$$\nu_1 := \nu_1(\omega) = 1, \qquad \omega \in (0, 2). \tag{3.11}$$

For $p \geqslant 5$, it is given by the union of the two arcs ν_1 and ν_1', where

$$\nu_1 := \nu_1(\omega) = 1, \qquad \omega \in (0, \omega^*], \tag{3.12}$$

and

$$\nu_1' := \nu_1'(\omega) = [F(\omega, \theta)]^{1/p}, \qquad \omega \in (\omega^*, 2), \tag{3.13}$$

with $\theta \in \theta^+ \backslash \{0\}$ being the solution to (2.8).

4. THE NONPOSITIVE CASE

As in Section 3, we try to find if a $\theta \in \theta^- \backslash \{\pi\}$ exists satisfying (2.9). From [7], for any $\omega \in [1, \hat{\omega}]$, $\theta = \pi$ is the only element of θ^-. So the right boundary of the convergence domain is

$$\nu_2(\omega) := \frac{1 + (1-\omega)^2}{(2-\omega)^{2/p}\omega^{2-2/p}}, \qquad \omega \in [1, \hat{\omega}]. \tag{4.1}$$

By Theorem 2.1, for any $\omega \in (\omega^{**}, 1)$ the only real nonpositive value of $F(\omega, \theta)$ is $F(\omega, \pi)$. This is because $a(\omega, \theta)$ strictly increases. Therefore the right boundary will be given again by

$$\nu_2(\omega) := \frac{1 + (1-\omega)^2}{(2-\omega)^{2/p}\omega^{2-2/p}}, \qquad \omega \in [\omega^{**}, 1]. \tag{4.2}$$

To proceed, for a fixed $\omega \in (0, \omega^{**})$, we recall from Theorem 2.1 that there exists a $\theta_0 \in [0, \pi]$ corresponding to the maximum value of $a(\omega, \theta) > \pi$. So there will exist a $\theta \in (0, \theta_0]$ which will satisfy (2.7), (2.9). Since, by Theorem 2.2, $r(\omega, \theta)$ strictly increases with θ, it will be $F(\omega, \pi) < F(\omega, \theta) < 0$. In case there are more than one $\theta \in \theta^{-}\backslash\{\pi\}$ satisfying (2.9), the smallest one, let it be θ_m, will give the right boundary. In other words,

$$\nu_2''(\omega) := [-F(\omega, \theta_m)]^{1/p}, \qquad \omega \in (0, \omega^{**}). \tag{4.3}$$

For $\omega > 1$ the case $\omega \in (\hat{\omega}, 2)$ is to be studied. The two lemmas below facilitate the analysis.

LEMMA 4.1. *For all* $11 \leqslant p \leqslant 30$, $a_1(\omega, \theta_0)$ *is a strictly decreasing function of* $y \in [\hat{y}, -2)$, *where* θ_0 *and* $\hat{y}$ *are given by* (2.19) *and* (2.28), *respectively. Moreover*

$$\tan a_1(\omega, \theta_0) = \frac{[y(y-2)(p-y)(y+p-2)]^{1/2}}{y^2 - 2y + p}. \tag{4.4}$$

Proof. See Lemma 4.1 of [9]. ∎

LEMMA 4.2. *For all* $p \geqslant 3$, $a_2(\omega, \theta_0)$ *is a strictly decreasing function of* y *for all* $y \in [\hat{y} - 2)$. *Moreover,*

$$\tan a_2(\omega, \theta_0) = -\frac{[(p-y)(y+p-2)]^{1/2}}{(p-1)(y^2-2y)^{1/2}}. \tag{4.5}$$

Proof. See Lemma 4.2 of [9]. ∎

One of our main results is given in the following statement.

THEOREM 4.3.

(i) *For any* $3 \leqslant p \leqslant 14$ *and a fixed* $\omega \in (\hat{\omega}, 2)$ *there exists a unique real negative value of* $F(\omega, \theta)$ *satisfying* (2.7) *and corresponding to* $\theta = \pi$.

(ii) *For* $p \geqslant 15$, *there exists a* $\tilde{y}$ *such that for any fixed* $y \in [\tilde{y}, -2)$ *there is at least one real negative value of* $F(\omega, \theta) \neq F(\omega, \pi)$.

Proof. (i): For any $3 \leqslant p \leqslant 11$, by virtue of Lemma 4.2,

$$a_2(\omega, \theta_0) > \lim_{y \to -2^-} a_2(\omega, \theta_0) = \arctan\left(- \frac{\sqrt{2}\,(p+2)^{1/2}(p-4)^{1/2}}{4(p-1)} \right). \tag{4.6}$$

By direct computation it can be obtained that $(p-2)\lim_{y \to -2^-} a_2(\omega, \theta_0) > -\pi$. Since $a_1(\omega, \theta_0) > 0$, we have $a(\omega, \theta_0) > -\pi$, implying that there is no value of θ other than $\theta = \pi$ for which (2.7) holds true. For $p = 12, 13, 14$, using Lemmas 4.1 and 4.2, it can be obtained computationally that

$$\min a(\omega, \theta_0) = \arctan\left(\frac{2\sqrt{2}\,(p+2)^{1/2}(p-4)^{1/2}}{p+8} \right)$$
$$+ (p-2)\arctan\left(- \frac{\sqrt{2}\,(p+2)^{1/2}(p-4)^{1/2}}{4(p-1)} \right) > -\pi.$$

In other words, the same conclusion as before holds.

(ii): As in the analysis of the nonnegative case, we study the sequences of values $a_1(\hat{\omega}, \hat{\theta}_0)$, $a_2(\hat{\omega}, \hat{\theta}_0)$, and $a(\hat{\omega}, \hat{\theta}_0)$ corresponding to $\hat{y}$, given by (2.28), as functions of p. From Lemmas 4.1 and 4.2, $a_1(\hat{\omega}, \hat{\theta}_0)$ is a strictly decreasing function of p for $11 \leqslant p \leqslant 30$, while $a_2(\hat{\omega}, \hat{\theta}_0)$ is a strictly decreasing function for all p. This is because, $\hat{y}$ strictly increases with p and $\lim_{p \to \infty} \hat{y} = -2$. Therefore $a(\hat{\omega}, \hat{\theta}_0)$, as a function of p, strictly decreases for $11 \leqslant p \leqslant 30$. Computationally, it can be found out that

$$a(\hat{\omega}, \hat{\theta}_0)|_{p=15} \approx -2.985 > -\pi > a(\hat{\omega}, \hat{\theta}_0)|_{p=16} \approx -3.311. \tag{4.7}$$

This result implies that for all $16 \leqslant p \leqslant 30$ and for all $y \in [\hat{y}, -2)$ it will hold that $a(\omega, \theta_0) < -\pi$. Hence, there exists $\tilde{y} \in (y^*, \hat{y}]$ such that (2.7) will be satisfied for more than one $\theta \in \theta^-$ for any fixed $y \in [\tilde{y}, -2)$. On the other hand, $a_1(\hat{\omega}, \hat{\theta}_0)|_{p \geqslant 3} \in (0, \pi)$, while $(p-2)a_2(\hat{\omega}, \hat{\theta}_0)|_{p=21} \approx -6.090 > -2\pi > (p-2)a_2(\hat{\omega}, \hat{\theta}_0)|_{p=22} \approx -6.432$. Therefore $a(\hat{\omega}, \hat{\theta}_0)|_{p \geqslant 22} < -\pi$. Consequently the same conclusion as before holds for any $p \geqslant 30$. For $p = 15$, it can be checked that $\min a(\omega, \theta_0) < -\pi$, meaning that there exists $\tilde{y} \in (\hat{y}, -2)$ such that there are more than one $\theta \in \theta^-$ for any fixed $y \in [\tilde{y}, -2)$. This completes our proof. ■

From Theorem 4.3 (i) it is concluded that the right boundary for $3 \leqslant p \leqslant 14$ and for all $\omega \in (1, 2)$ will be given by the formula (4.1). A typical region of convergence is illustrated in Figure 3.

For $p \geqslant 16$ and for a fixed $y \in [\tilde{y}, \hat{y}]$, Theorem 2.4 states that the largest real negative value of $F(\omega, \theta)$ is $F(\omega, \pi) = -r(\omega, \pi)$. From (2.26) this value is given by

$$F(\omega, \pi) = -\frac{(-y)^p}{(2-y)^{p-1}(-y-2)}. \tag{4.8}$$

Differentiating the above expression w.r.t. y, it can be proved that it is a strictly decreasing function for all $y \geqslant -2p/(p-2)$. Since $\hat{y} > -2p/(p-2)$, it is concluded that $F(\omega, \pi)$ strictly decreases for $y \in [\hat{y}, -2)$, with $\lim_{y \to -2^-} F(\omega, \pi) = -\infty$. Based on continuity arguments, we can say that the above value, $F(\omega, \pi)$, must be the largest one in an interval of y whose right endpoint $y' > \hat{y}$. Then it is concluded that for $y \in (y', -2)$ the largest real negative value $F(\omega, \theta)$ satisfying (2.9) will become greater than $F(\omega, \pi)$.

Summarizing the conclusions so far, we have that for $y \leqslant y'$ the right boundary of the convergence domain will be given by $\nu_2(\omega)$ of (4.1), while for $y > y'$ there will exist a right boundary, other than $\nu_2(\omega)$, corresponding to the solution of (2.7), (2.9).

In the previous analysis the case $p = 15$ was not covered. This is done using the lemma below.

LEMMA 4.4. *The function* $r(\omega, \theta_0)$ *which is given by*

$$r(\omega, \theta_0) = \frac{p^{p/2}}{(p-2)^{p/2-1}} \left(\frac{(-y)}{2-y} \right)^{p/2} (1-y) \tag{4.9}$$

is a strictly decreasing function w.r.t. $y \in (y^*, -2)$.

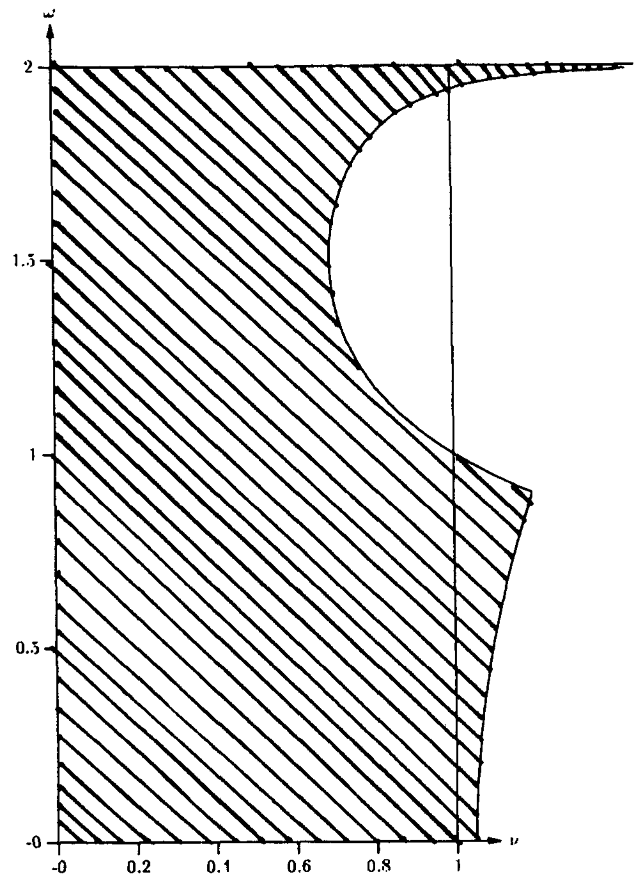

FIG. 3. Nonpositive case ($3 \leqslant p \leqslant 14$).

Proof. A direct substitution of (2.19) in (2.26) yields (4.9). Since both $-y/(2-y)$ and $1-y$ are positive and strictly decreasing functions of y, so is $r(\omega, \theta_0)$. ■

For $p = 15$, it is found computationally that for $y_1 = -2.0959$ and $y_2 = -2.0949$

$$a(\omega_1, \theta_0) = -3.1406 > -\pi > a(\omega_2, \theta_0) = -3.1421.$$

On the other hand, we can find out that

$$r(\omega_1, \theta_0) = 0.65519, \qquad r(\omega_2, \theta_0) = 0.65508,$$

$$r(\omega_1, \pi) = 0.431965, \qquad r(\omega_2, \pi) = 0.432354. \tag{4.10}$$

Since $r(\omega, \theta_0)$ strictly decreases while $r(\omega, \pi)$ strictly increases with y, it is implied from (4.10) that there will be a $\tilde{y} \in (-2.0959, -2.0949)$ such that $F(\tilde{\omega}, \theta_0) \in (-0.65519, -0.65508)$ and $F(\tilde{\omega}, \pi) \in (-0.431965, -0.432354)$. Consequently, $F(\tilde{\omega}, \theta_0) < F(\tilde{\omega}, \pi)$. the rest of the argumentation is that of the case $p \geqslant 16$, implying that for $p = 15$ exactly the same conclusion holds.

Therefore for all $p \geqslant 15$ and for any $\omega \in (\omega', 2)$ the right boundary will be given by an expression of the form

$$\nu_2''(\omega) := [-F(\omega, \theta)]^{1/p}, \tag{4.11}$$

with $\theta \in \theta^{-} \backslash \{\pi\}$ being the solution to (2.9).

A typical convergence domain for $p \geqslant 15$ is illustrated in Figure 4.

5. FINAL REMARKS AND PARTICULAR CASES

The analysis so far has allowed us to determine the exact convergence domains for the block SSOR method when the corresponding block Jacobi matrix B (or its transpose) is weakly cyclic of index $p \geqslant 3$. This was done in the two cases of $\sigma(B^p)$ nonnegative or nonpositive. It is recalled that except for those parts of the arcs of the right boundaries of the convergence domains that were known (see [7]) or are extensions of the known ones, the remaining parts can be determined through (2.6), (2.8) [or (2.7), (2.9)]. It is noted that analytic expressions for $\cos\theta$, $\theta \in (0, \pi)$, can only be found of $p = 3, 4, 5$, and 6. In all other nontrivial cases, for each $p \geqslant 7$ and each ω, $\cos\theta$ has to be found computationally. Consequently, the same holds true for the corresponding parts of the right boundaries.

In what follows we work out the cases $p = 3$ and 4 for $\sigma(B^p)$ nonpositive, since the corresponding nonnegative cases have already been examined in Section 3.

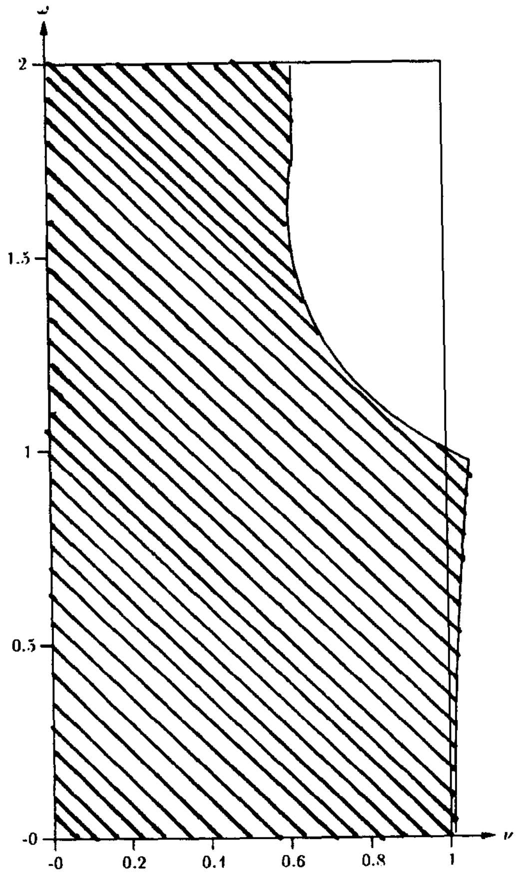

FIG. 4. Nonpositive case ($p \geqslant 15$).

$p = 3$. From (2.7) and (2.9) by using (2.1) we can take

$$\cos\theta = -\frac{y^3 - y^2 - 2y - 2}{2(y^2 - y - 1)}, \qquad \omega \in (0, \omega_3^{**}), \quad \omega_3^{**} = \frac{-1 + \sqrt{5}}{2} \tag{5.1}$$

(the golden section number). So, using (5.1) in (2.3) and then in (2.1), it can be obtained that

$$\nu_2''(\omega) := \frac{\left[(1-\omega)^2+1\right](2-\omega)^{1/3}}{(1-\omega)^{1/3}\left[(1-\omega)^5+1\right]^{1/3}}, \qquad \omega \in (0, \omega_3^{**}). \quad (5.2)$$

It is interesting to point out that $\lim_{\omega \to 0^+} \nu_2''(\omega) = 2$. It is noted that the convergence domain $R^-(3)$ is the *only* convergence domain whose arc of the right boundary for $\omega \in (0, \omega_3^{**})$ lies strictly to the right of the line $\nu = \nu_2''(\omega_3^{**})$ and *not* to the left of it as is illustrated in Figure 3.

$p = 4$. This time it is found that

$$\cos\theta = \frac{-y^2+2y+2}{2(y-1)}, \qquad \omega \in (0, \omega_4^{**}), \quad \omega_4^{**} = -1+\sqrt{3}. \quad (5.3)$$

From (5.3) and (2.3), (2.1) it can be obtained that

$$\nu_2''(\omega) := \frac{\left[(1-\omega)^2+1\right]^{1/2}}{(1-\omega)^{1/4}\left[(1-\omega)^2-(1-\omega)+1\right]^{1/4}}, \qquad \omega \in (0, \omega_4^{**}). \quad (5.4)$$

On the other hand we have $\lim_{\omega \to 0^+} \nu_2''(\omega) = \sqrt{2}$.

Finally, we report that we have worked out the case $p = 5$, computationally, by using Sturm sequences [10]. The results obtained confirm the theoretical ones in Section 4.

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