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# EXACT SOR CONVERGENCE REGIONS FOR A GENERAL CLASS OF P-CYCLIC MATRICES *†

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**Abstract.**

Linear systems whose associated block Jacobi iteration matrix $B$ is weakly cyclic generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ in the spirit of Li and Varga are considered. Regions of convergence for the corresponding block $p$-cyclic SOR method are derived and the exact convergence domains for real spectra, $\sigma(B^p)$, of the same sign are obtained. Moreover, analytical expressions for two special cases for $p = 5$ are given and numerical results are presented confirming the theory developed. The tools used for this work are mainly from complex analysis and extensive use of (asteroidal) hypocycloids in the complex plane is made to produce our results.

*AMS subject classification:* 65F20.

*Key words:* p-cyclic matrices, SOR method, hypocycloids.

## 1 Introduction and preliminaries.

Suppose that $A \in \mathbb{C}^{n,n}$ is partitioned in a $p \times p$ block form and its diagonal blocks $A_{ii}$, $i = 1(1)p$, are square and nonsingular. Suppose also that the associated block Jacobi iteration matrix

$$B := I - D^{-1}A, \tag{1.1}$$

with $D := diag(A_{11}, A_{22}, \ldots, A_{pp})$, is weakly cyclic generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ in the spirit of Li and Varga [16]. According to their definition: "*The $p \times p$ block matrix $B$ is a weakly cyclic matrix generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ of the integers $\{1, 2, \ldots, p\}$ iff*

$$B_{\sigma_j \sigma_{j+1}} \not\equiv 0, \quad j = 1(1)p, \quad \sigma_{p+1} = \sigma_1, \tag{1.2}$$

*and*

$$B_{ij} \equiv 0 \quad \textit{otherwise.}" \tag{1.3}$$

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We note here that the block Jacobi matrices associated with a class of block $p$-cyclic consistently ordered matrices [27, 31, 1] are generated, according to the above definition, by $\sigma = (p, p-1, p-2, \ldots, 1)$ while those of a class of the $(q, p-q)$-generalized consistently ordered matrices (or $(q, p-q)$-GCO) (see [31]) are generated by $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ with $1 \leq \sigma_j \leq p$, $j = 1(1)p$, and $\sigma_{j+1} = p - q + \sigma_j$ or $\sigma_{j+1} = \sigma_j - q$. So, the definition presented above is the most general one for the family of block $p$-cyclic matrices which have only $p$ nonidentically zero blocks and diagonal blocks square nullmatrices. Obviously, the block graph of the block matrix $B$ is a cycle.

For the solution of the nonsingular linear system

$$Ax = b \tag{1.4}$$

the Successive Overrelaxation (SOR) iterative method

$$x^{(m+1)} = \mathcal{L}_\omega x^{(m)} + c_\omega, \qquad m = 0, 1, 2, \ldots \tag{1.5}$$

is considered, where $x^{(0)} \in \mathbb{C}^n$ is arbitrary, $\omega$ is the relaxation factor and

$$\begin{aligned} \mathcal{L}_\omega &:= (I - \omega L)^{-1}[(1-\omega)I + \omega U], \\ c_\omega &:= \omega(I - \omega L)^{-1} D^{-1} b \end{aligned} \tag{1.6}$$

with $L$ and $U$ being the strictly lower and the strictly upper triangular components of the Jacobi matrix $B$, respectively.

In case A belongs to the family of matrices mentioned previously the sets of the eigenvalues $\mu \in \sigma(B)$ and $\lambda \in \sigma(\mathcal{L}_\omega)$ are connected via the functional equation

$$(\lambda + \omega - 1)^p = \omega^p \mu^p \lambda^{|\zeta_L|}. \tag{1.7}$$

Equation (1.7) is a special case of the functional equation

$$\begin{aligned} [\lambda - (1-\omega)(1-\hat{\omega})]^p &= \lambda^k [\lambda\omega + \hat{\omega} - \omega\hat{\omega}]^{|\zeta_L| - k} \times \\ &\quad [\lambda\hat{\omega} + \omega - \omega\hat{\omega}]^{|\zeta_U| - k} (\omega + \hat{\omega} - \omega\hat{\omega})^{2k} \mu^p \end{aligned} \tag{1.8}$$

which connects the eigenvalues $\mu$ of $B$ and $\lambda$ of the Unsymmetric Successive Overrelaxation (USSOR) iteration operator $T_{\omega,\hat{\omega}}$. Equation (1.8) is due to Li and Varga [16]. In (1.8), $|\zeta_L|$, $|\zeta_U|$ and $k$ are integers which are well-defined in [16] (see also [23]). More specifically, from the definitions in [16], $|\zeta_L|$ and $|\zeta_U|$ are the numbers of the nonzero blocks in the block triangular matrices $L$ and $U$ while $k$ is the number of nonzero blocks of the matrix product $LU$. Note that (1.7) is derived from (1.8) for $\hat{\omega} = 0$, with $\hat{\omega}$ being the underrelaxation parameter.

The functional equation (1.7) generalizes the one for $(q, p-q)$-GCO matrices of Verner and Bernal [28], which was first mentioned by Varga (see [27]). It is derived from (1.7) for $|\zeta_L| = p - q$. In this work we study equation (1.7), where, without loss of generality, we assume that $|\zeta_L| = p - q$ and that g.c.d.$(p, q) = 1$.

After the most recent work on the best block $p-$cyclic repartitioning by Markham, Neumann and Plemmons [17], Pierce, Hadjidimos and Plemmons [24], Eiermann, Niethammer and Ruttan [3] and Galanis and Hadjidimos [4] and the work on the solution of large scale systems arising in queuing network problems in Markov analysis, which plays an increasing role in computer, communication and transportation systems, by Kontovasilis, Plemmons and Stewart [14] and Hadjidimos and Plemmons [12, 13] the study of the convergence properties of the $p-$cyclic SOR has become more demanding. More specifically for the study of the aforementioned convergence properties, in the case we are interested in, information about the spectrum of $B$, $\sigma(B)$, may enable us to answer one or more of the following questions:

(i) What is the largest region, in the complex plane, containing $\sigma(B)$ for which (1.4) converges?

(ii) For what pairs $(\rho(B), \omega)$ does (1.4) converge? and

(iii) What is the (optimal) value of $\omega$ that minimizes $\rho(\mathcal{L}_\omega)$ for a given $\rho(B)$?

Complete answers to the three questions above have only been given for very particular classes of $p$-cyclic matrices. Answers to question (i) have been given for the class of block $p$-cyclic consistently ordered matrices, among others, by Young [30], (see also [31]), Varga [26], (see also [27]), Niethammer and Varga [20], Galanis, Hadjidimos, and Noutsos [5, 6, 7], Wild and Niethammer [29], Eiermann, Niethammer, and Ruttan [3], Kredell [15], Niethammer [19], Kontovasilis, Plemmons, and Stewart [14], Noutsos [22], and Hadjidimos and Plemmons [13]. Answers to question (ii) have again been given for the block $p$-cyclic consistently ordered matrices by many researchers (see, e.g., [30], [10], for nonnegative spectra $\sigma(B^p)$, as well as [15, 19, 21, 5, 6, 7, 29], for nonpositive spectra $\sigma(B^p)$ and [10, 29, 22] for both nonnegative and nonpositive spectra). For block $(q, p-q)$-GCO matrices $q \neq 1$ by Nichols and Fox [18] for nonnegative spectra, by Galanis, Hadjidimos, Noutsos and Tzoumas [8] for nonpositive spectra in the case of $(p-1, 1)$-GCO matrices and by Hadjidimos, Noutsos and Tzoumas [11] for both cases. Finally, answers to question (iii) for the block $p$-cyclic consistently ordered matrices have been given in [30, 15, 19], for $p = 2$, in [21] for $p = 3$, and in [26, 7, 29], and other works, for any $p$.

In the present work we shall try to give an answer to the first two questions raised for the general case of the family of matrices $A$ considered in the beginning. We organize our work as follows. In Section 2 the answer to question (i) is given, where among the well-known hypocycloidal curves used in the analysis asteroidal ones are also considered. In Section 3 we study the domains of convergence in the $(\rho(B), \omega)$-plane. Finally, in Section 4 we study in detail the particular cases $(p, q) = (5, 2)$ and $(5, 3)$. As will be seen some new results are obtained and some known ones are recovered. We conclude our study by presenting and working out numerical examples from Markov chain analysis. As will be seen the numerical data obtained confirm the theoretical findings of previous works as well as of the present one.

## 2 Hypocycloids and regions of convergence.

We begin our analysis with the functional eigenvalue relationship (1.7) by replacing $|\zeta_L|$, with $p-q$, and we consider that $\omega \in (1,2)$. The analysis for the case $\omega \in (0,1)$ is similar. Using the transformation

$$\phi = \frac{1}{\lambda} \tag{2.1}$$

and substituting in (1.7) we obtain

$$(1-(1-\omega)\phi)^p = \omega^p \mu^p \phi^q. \tag{2.2}$$

Our objective is to find the smallest region in the complex plane containing the eigenvalues $\mu \in \sigma(B)$ and which has an image, through the mapping (2.2), in the exterior of the circle $\partial D_\eta$ (where $D_\eta := \{\phi : \ \phi = \eta e^{i\theta}, \eta > 0, \ \ \theta \in [0, 2\pi)\}$, or, equivalently, in the interior of $\partial D_{1/\eta}$ since $\lambda = \frac{1}{\eta} e^{-i\theta}$. Then the spectral radius of the SOR iteration matrix, $\rho(\mathcal{L}_\omega)$, will be less than or equal to $\frac{1}{\eta}$, with equality holding iff there is an eigenvalue of $B$ on the boundary of the region to be found.

Extracting $p^{th}$ roots of both sides of (2.2) we obtain

$$1-(1-\omega)\phi = \omega\mu(\phi^q)^{1/p} = \omega\mu\phi^{q/p}. \tag{2.3}$$

and the corresponding region for $\mu$ is defined by the transformation

$$z := \frac{1-(1-\omega)\phi}{\omega\phi^{q/p}}. \tag{2.4}$$

To study the mapping in (2.4) we use the transformation

$$\zeta := \phi^{q/p} \tag{2.5}$$

and then (2.4) becomes

$$z := z(\zeta) = \frac{1-(1-\omega)\zeta^{p/q}}{\omega\zeta}. \tag{2.6}$$

From (2.5) we have that the closed disk $\overline{D}_\eta$, $|\phi| \leq \eta$, is mapped onto the sectors

$$S_k = \{\rho^{q/p} e^{i(2k\pi+\theta)q/p} : \rho \in [0,\eta], \ \ \theta \in [0, 2\pi)\}, \qquad k = 0(1)p-1$$

or, equivalently, onto

$$S_k = \{\rho^{q/p} e^{i(\frac{2kq\pi}{p}+\theta)} : \rho \in [0,\eta], \ \ \theta \in [0, \frac{2q\pi}{p})\}, \qquad k = 0(1)p-1. \tag{2.7}$$

Consequently, we have to study the transformation (2.6) for each one of the above sectors. However, since from (2.6), if $\zeta' = \zeta e^{i\frac{2kq\pi}{p}}$, $k = 1(1)p-1$, then for the images $z'$ of $\zeta'$ and $z$ of $\zeta$ there holds $z' = z e^{-i\frac{2iq\pi}{p}}$, it suffices to study (2.6) for $S_0$ only. Let then $\overline{S}_0$ be the associated closed sector, i.e.,

$$\overline{S}_0 = \{\rho^{q/p} e^{i\theta} : \rho \in [0,\eta], \ \ \theta \in [0, \frac{2q\pi}{p}]\}. \tag{2.8}$$

Obviously, the boundary of $\overline{S}_0$ is the closed curve

$$\partial S_0 = \{\rho : \rho \in [0,\eta]\} \cup \{\rho e^{i\frac{2q\pi}{p}} : \rho \in [0,\eta]\} \cup \{\eta e^{i\theta} : \theta \in [0, \frac{2q\pi}{p}]\}. \tag{2.9}$$

The image of $\partial S_0$ by means of (2.6) is a closed curve that consists of the union of the images of the three segments in (2.9). Since the second line segment in (2.9) is obtained from the first one by a rotation about the origin through an angle $\frac{2q\pi}{p}$ it suffices to study the first line segment and the third curved one. For this we have:

LEMMA 2.1. *The image of the line segment $\{\rho : \rho \in [0,\eta]\}$ via (2.6) is an infinite line segment on the positive real semiaxis. Moreover, if $\eta \le \hat{\eta} := \left[\frac{q}{(p-q)(\omega-1)}\right]^{q/p}$ the mapping in question is a $1-1$ onto the line segment $\left[\frac{1-(1-\omega)\eta^{p/q}}{\omega\eta}, \infty\right]$. On the other hand if $\eta \ge \hat{\eta}$ the image is the line segment $\left[\frac{1-(1-\omega)\hat{\eta}^{p/q}}{\omega\hat{\eta}}, \infty\right]$ and the corresponding mapping is not a $1-1$ one.*

PROOF: Differentiating (2.6), for $\zeta = \rho$, with respect to (wrt) $\rho$, it is readily checked that $\frac{\partial z}{\partial \rho} < 0$ for $\rho < \hat{\eta}$, $\frac{\partial z}{\partial \rho} = 0$ at $\rho = \hat{\eta}$, and $\frac{\partial z}{\partial \rho} > 0$ for $\rho > \hat{\eta}$. Therefore, for $\rho$ increasing continuously from 0 to $\eta \le \hat{\eta}$ the corresponding images decrease continuously from $\infty$ to $\frac{1-(1-\omega)\eta^{p/q}}{\omega\eta}$ and the mapping is $1-1$. On the other hand, for $\eta > \hat{\eta}$ the images of $\rho$, increasing from 0 to $\hat{\eta}$, decrease from $\infty$ to $\frac{1-(1-\omega)\hat{\eta}^{p/q}}{\omega\hat{\eta}}$ and as $\rho$ keeps on increasing up to $\eta$, its images increase from $\frac{1-(1-\omega)\hat{\eta}^{p/q}}{\omega\hat{\eta}}$ to $\frac{1-(1-\omega)\eta^{p/q}}{\omega\eta}$. Hence the mapping is not $1-1$. □

To study the image of the third curved segment of (2.9) via (2.6) we consider the parametric equations of the curve. These are given by

$$\begin{aligned} x = Re z &= \frac{1}{\omega\eta^{q/p}}[\cos\theta - (1-\omega)\eta\cos\left((\tfrac{p-q}{q})\theta\right)] \\ y = Im z &= -\frac{1}{\omega\eta^{q/p}}[\sin\theta + (1-\omega)\eta\sin\left((\tfrac{p-q}{q})\theta\right)] \end{aligned} \qquad \theta \in [0, \frac{2q\pi}{p}]. \tag{2.10}$$

As is known equations (2.10) define a hypocycloidal curve. Equations (2.10) can be rewritten as follows

$$\begin{aligned} x &= (R-r)\cos t + h\cos\left((\tfrac{R-r}{r})t\right) \\ y &= (R-r)\sin t - h\sin\left((\tfrac{R-r}{r})t\right) \end{aligned} \qquad t \in [0, 2\pi], \tag{2.11}$$

where $R$ and $r$ are the radii of the large and of the small circle, respectively, and $h$ is the distance from the center of the small circle of the fixed point on the disk of the small circle when this circle rolls in the interior of the large one without sliding. From (2.10) and (2.11) it becomes clear that

$$(2.12) \qquad R-r=\frac{1}{\omega\eta^q}, \quad h=\frac{(\omega-1)\eta^{p-q}}{\omega}, \quad \frac{R-r}{r}=\frac{p-q}{q}.$$

A study of the transformation (2.6) using these hypocycloidal curves in a similar way as in [29] follows.

LEMMA 2.2. *The image of the arc* $\{\eta e^{i\theta} : \theta \in [0, \frac{2q\pi}{p}]\}$ *via (2.6) is the hypocycloidal curve (2.10). The mapping is* $1-1$ *and the curve obtained is symmetric with axis of symmetry the straight line* $\{\rho e^{-i\frac{q\pi}{p}} : \rho \in \mathbb{R}\}$.

PROOF: Using (2.6) it is easy to prove that the images of two symmetric points with respect to the axis $\{\rho e^{i\frac{q\pi}{p}} : \rho \in \mathbb{R}\}$ are symmetric points wrt $\rho e^{i\frac{q\pi}{p}}$. To prove that the mapping is $1-1$, consider two distinct points $\eta e^{i\theta_1}$, $\eta e^{i\theta_2}$ with $\theta_1, \theta_2 \in (0, \frac{2q\pi}{p})$. Let that the images of these two points coincide. From (2.10), after some operations take place, we obtain that $\sin(\frac{p}{2q}(\theta_1+\theta_2)) = 0$ or $\theta_1+\theta_2 = \frac{2kq\pi}{p}$. Since $\theta_1, \theta_2 \in (0, \frac{2q\pi}{p})$, $k=1$. Also, since $\theta_1$, $\theta_2$ are distinct it is implied that the images are distinct and symmetric wrt the axis $\{\rho e^{iq\pi}, \rho \in \mathbb{R}\}$. This conclusion contradicts our assumption that the images of the two points coincide. Consequently, the mapping is $1-1$. □

We distinguish the two cases $\frac{p-q}{q} < 1$ and $\frac{p-q}{q} > 1$. The first inequality implies that the number of nonidentically zero blocks $|\zeta_L|$ of the matrix $L$ is less than that of $U$, $|\zeta_U|$, while in the second case the situation is reversed. On the other hand, from (2.12), $\frac{p-q}{q} < 1$ implies $R < 2r$ while $\frac{p-q}{q} > 1$ implies $R > 2r$.

After the analysis done so far we give in Figure 2.1 the various shapes of the corresponding hypocycloidal arcs in all possible cases.

Based on the previous analysis the following theorem can be stated and proved.

THEOREM 2.3. *The sector* $S_0' := S_0 \setminus \{\rho : \rho \in (0, \eta]\}$ *is mapped via (2.6) onto the set of points* $R_0' = z(S_0')$ *which is an open sector of the complex plane described by a semiline by a rotation about the origin through an angle of* $-\frac{2q\pi}{p}$ *and has as a boundary the image of the curve* $\partial S_0$ *via (2.6). Furthermore, the mapping in question is* $1-1$.

PROOF: First we define the set $R_0 := z(S_0)$ and note that $S_0'$, from which $R_0'$ was defined, is nothing but $S_0$ from which the boundary line segments are excluded. This is done because, in view of Lemma 2.1, the mapping for $\eta > \hat{\eta}$ is not $1-1$. Consider then the point $\zeta_0 = \rho_0 e^{i\theta_0} \in S_0'$. This point can be defined as the intersection of the semiline $\{\rho e^{i\theta_0} : \rho > 0\}$ and of the arc $\{\rho_0 e^{i\theta} : \theta \in (0, \frac{2q\pi}{p})\}$. By virtue of Lemma 2.2 we have that the mapping of the arc in question is a $1-1$ one. To find out that the mapping of the semiline in question is also $1-1$ we assume that it is not. This means that there are at least two distinct points of it that have the same image. Let these points be $\zeta_1 = \rho_1 e^{i\theta_0}$ and $\zeta_2 = \rho_2 e^{i\theta_0}$. Equating the expressions for their images, via (2.6), we have that

$$(2.13) \qquad \rho_2 - \rho_1 = (\omega-1)(\rho_2^{p/q} - \rho_1^{p/q})e^{ip/q\theta_0}.$$

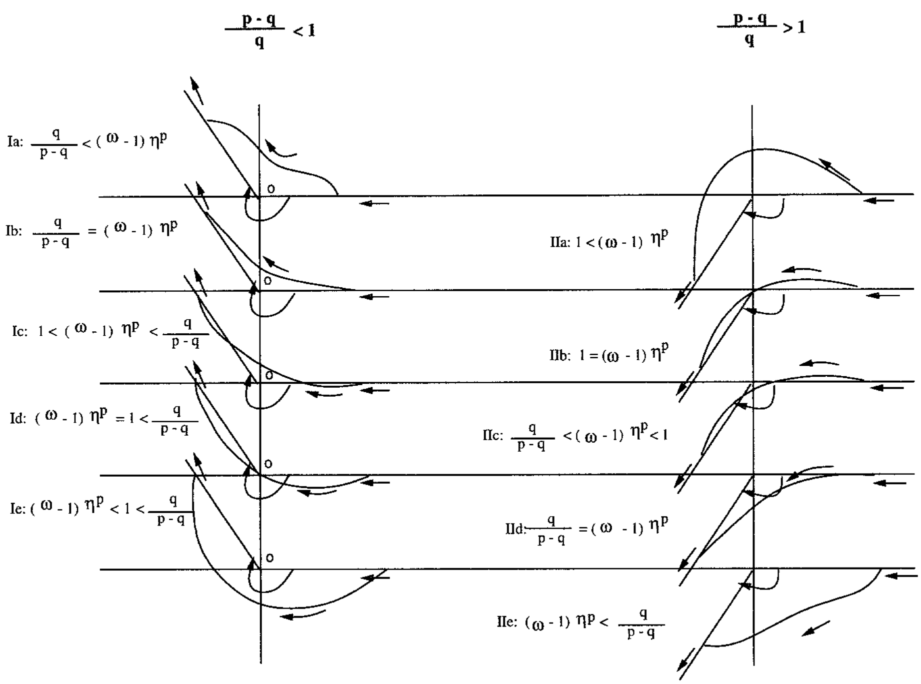



Figure 2.1: Hypocycloidal arcs in all possible cases.

If $\rho_1 \neq \rho_2$, in (2.13), then we will have equality between a real and a complex number unless $\theta_0 = \frac{\pi q}{p}$ in which case the two members of (2.13) will be real of opposite signs. Hence (2.13) holds iff $\rho_1 = \rho_2$ implying that the mapping is $1-1$. We have then proved that the image of the point $\zeta_0$ is the unique intersection point $z_0$ of the images of the two lines considered. Since this holds for any point $\zeta_0 \in S_0'$ the mapping is $1-1$. Moreover, it is readily checked that $R_0$ is the left-hand-side part of the complex plane wrt the direction of the arrows, as shown in Figure 2.1, and the proof is complete. □

It is also immediately seen, using induction, that the conclusions of Theorem 2.3 hold also for all sectors $S_k'$, $k = 0(1)p-1$, in (2.7). This is because for the mappings $R_k := z(S_k')$, $k = 0(1)p-1$, there holds $R_k = R_{k-1}e^{-i\frac{2q\pi}{p}}$, $k = 1(1)p-1$, or

$$R_k = R_0 e^{-i\frac{2kq\pi}{p}}, \qquad k = 1(1)p-1. \tag{2.14}$$

From the analysis so far it is also clear that since the disk $\overline{D}_{\eta^{q/p}}$, $|\zeta| \leq \eta^{q/p}$, is the union of the sectors $S_k$, $k = 0(1)p-1$, it will be mapped through (2.6) into $\bigcup_{k=0}^{p-1} R_k$. Consequently, all the points belonging to $\left(\bigcup_{k=0}^{p-1} R_k\right)^c = \bigcap_{k=0}^{p-1} R_k^c$ will be images only of points belonging to the exterior of the disk $\overline{D}_{\eta^{q/p}}$.

Based on the analysis so far the theorem below can be stated and proved.

THEOREM 2.4. *Let the Jacobi matrix $B$ associated with the linear system (2.1) be block weakly cyclic of index $p$ generated by the cyclic permutation $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_p\}$ with $|\zeta_L| = p - q$. Then for the spectral radius of the corresponding SOR iteration matrix there holds $\rho(\mathcal{L}_\omega) \le \frac{1}{\eta}$ iff $\sigma(B) \subseteq \bigcap_{k=0}^{p-1} R_k^c$, with $R_k$ being defined in (2.14), $R_0 = z(S_0)$, and $z$ and $S_0$ in (2.6) and (2.7), respectively. Moreover, $\rho(\mathcal{L}_\omega) = \frac{1}{\eta}$ iff at least one element of $\sigma(B)$ lies on the boundary of $\bigcap_{k=0}^{p-1} R_k^c$.*

PROOF: If all the eigenvalues of $B$ belong to the region defined above then from the successive transformations (2.1), (2.5), (2.6) and the analysis done so far it follows that all the eigenvalues of $\mathcal{L}_\omega$ belong to the closed disk $\overline{D}_{1/\eta}$ with radius $\frac{1}{\eta}$. Conversely, if there exists an eigenvalue $\mu \in \sigma(B)$ such that $\mu \in \bigcup_{k=0}^{p-1} R_k$, then there will exist $i \in \{0, 1, 2, \ldots, p-1\}$ such that $\mu \in R_i$ and from the previous transformations there exists $\lambda \in \sigma(\mathcal{L}_\omega)$ such that $|\lambda| > \frac{1}{\eta}$ and the proof is complete. □

REMARK: We simply remark that in view of Theorem 2.4 and the various cases illustrated in Figure 2.1, the region of convergence $\bigcap_{k=0}^{p-1} R_k^c$ is the empty set $\emptyset$ in the cases $Ia$, $Ib$, $Ic$, $Id$, $IIa$ and $IIb$. So, the cases to be studied in the sequel are the remaining ones of Figure 2.1 or, equivalently, the ones where

$$\eta < \hat{\eta} = \frac{1}{(\omega - 1)^{\frac{1}{p}}}. \tag{2.15}$$

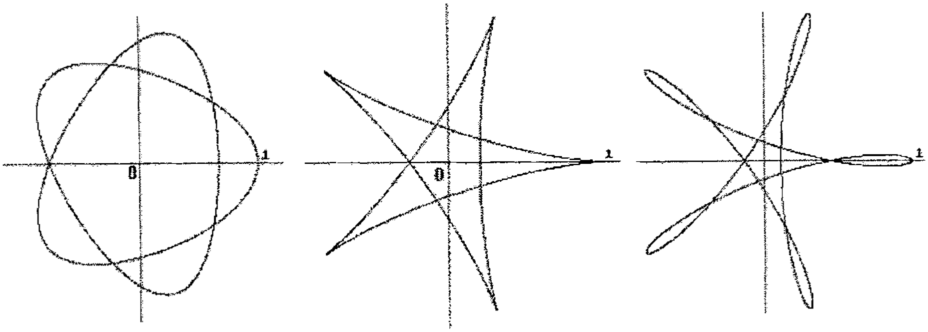
The regions of convergence for the cases $(p, q) = (5, 2)$ and $(5, 3)$ are illustrated in Figures 2.2 and 2.3, respectively.

Before we close this section we mention that in the case where $\omega \in (0, 1)$ we begin with the transformation (2.6) and follow an analysis analogous to the one in [29], the corresponding regions of convergence can be determined from those obtained for $\omega \in (1, 2)$ by a rotation about the origin through an angle $\frac{q}{p}\pi$. Analogous conclusions to the ones obtained so far can be drawn which are not presented here.

## 3 Domains of convergence in the $(\rho(B), \omega)$-plane.

From the previous analysis it has become obvious that the region of convergence remains the same by a rotation about the origin through an angle $\frac{2q}{p}\pi$ or $\frac{2kq}{p}\pi$, $k = 0(1)p-1$. Since g.c.d.$(p, q) = 1$, the endpoints of the arcs defining the region of convergence will be $p$ distinct points of a circle separating the circle into $p$ equal consecutive arcs. Hence, rotation of the region of convergence through an angle $\frac{2k\pi}{p}$, $k = 0(1)p - 1$, about the origin will give the same region.

In what follows we study the behavior of the length $r$ of the polar radius from the origin to a point of the boundary of the region of convergence as a function of the polar angle $\theta$. Thus, from the parametric equations (2.10) we have




(a) (c) (b)

Figure 2.2: Region of convergence for the case $(p,q) = (5,2)$.

(3.1) $r := |z(\zeta)| = \frac{1}{\omega\eta^{p/q}}[1 + (1-\omega)^2\eta^2 - 2(1-\omega)\eta\cos\frac{p}{q}\theta]^{1/2}, \quad \omega \in (1,2).$

Obviously, $r$ as a function of $\theta$ is a decreasing one in $[0, \frac{q}{p}\pi]$ and an increasing one in $[\frac{q}{p}\pi, \frac{2q}{p}\pi]$. So, the smallest value of $r$, $r_{min}$, is assumed for $\theta = \frac{q\pi}{p}$ and is given by

$$r_{min} = \frac{1}{\omega\eta^{p/q}}[1 + (1-\omega)\eta]. \tag{3.2}$$

Because of the cyclic nature of the boundary (curve) of the region of convergence

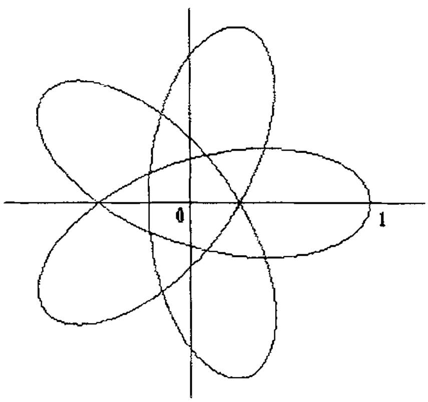



Figure 2.3: Regions of convergence for the case $(p,q) = (5,3)$.

all points of the boundary that have polar angles $\theta = \frac{(2k+1)q}{p}\pi$, $k = 0(1)p-1$, will have polar radii equal to $r_{min}$. These arcs correspond to one of the sets of arcs $\{\frac{2k\pi}{p}, k = 0(1)p-1\}$ or $\{\frac{2k+1}{p}\pi, k = 0(1)p-1\}$. To the other one there will correspond the points of the region's boundary with the maximum absolute value, let it be $r_{max}$. To find this value we have to find the point of intersection of the boundary curve with the real axis. $r_{min}$ will be the polar radius of the intersection point with the positive semiaxis iff the difference $0 - q\pi/p$ is an integral multiple of $2\pi/p$, or, equivalently, if $q$ is even otherwise the polar radius on the positive semiaxis will be $r_{max}$. Similarly, $r_{min}$ will correspond to the intersection of the boundary curve with the negative semiaxis if $\pi - \frac{q\pi}{p}$ is an integral multiple of $\frac{2\pi}{p}$ or if $p - q$ is even. Otherwise the intersection with the negative semiaxis will have polar radius $r_{max}$. We distinguish then three cases:

i) $p$ even $q$ odd. On both semiaxes the corresponding polar radii will be equal to $r_{max}$ for $\omega > 1$ and equal to $r_{min}$ for $\omega < 1$.

ii) $p$ odd $q$ even. To the positive semiaxis will correspond a polar radius equal to $r_{min}$ and to the negative one $r_{max}$.

iii) $p$ and $q$ odd. To the positive semiaxis will correspond $r_{max}$ and to the negative one $r_{min}$ for $\omega > 1$ while for $\omega < 1$ the situation is reversed.

To conclude the analysis regarding the region of convergence, $r_{max}$ must be determined. For this we begin with (2.10), where in order to find the points of intersection with the real axis we put $y = 0$ to obtain

$$\sin\theta + (1-\omega)\eta\sin\left(\frac{p-q}{p}\theta\right) = 0, \qquad \theta \in [0, 2\pi) \tag{3.3}$$

or, equivalently,

$$U_{q-1}(t) + (1-\omega)\eta U_{p-q-1}(t) = 0, \tag{3.4}$$

with $t = \cos\frac{\theta}{p}$ and $U_s(t)$ being the Chebyshev polynomial of the second kind of degree $s$ [25]. The intersection points are found from the first equation of (2.10) using the expressions

$$x_i = \frac{1}{\omega\eta^{q/p}}[T_q(t_i) - (1-\omega)\eta T_{p-q}(t_i)], \quad i = 0(1)p-2 \tag{3.5}$$

with $t_i$ being the zeros of (3.4) and $T_s(t)$ the Chebyshev polynomial of the first kind of degree $s$. Then $r_{max}$ will be the smallest value of $x_i$ in cases (i) and (iii) and/or the absolute value of the maximum values of the negative $x_i$'s in cases (i) and (ii). Apparently, in the general case, this value can only be given numerically.

Here we simply note that in case $r_{max}$ does not belong to the real axis then we rotate the right hand side of (2.6) about the origin through an angle equal to $\frac{q\pi}{p}$ and work in a similar way as in the one we described above.

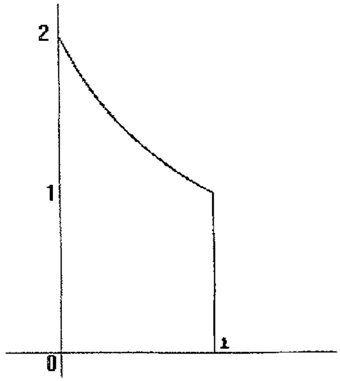



Figure 3.1: Convergence domain in the $(\rho(B), \omega)$-plane.

In the remaining part of this section we examine in some detail the two cases of $\sigma(B^p)$ being nonnegative and nonpositive.

The first statements that can be proved are the following ones:

THEOREM 3.1. *Let $\sigma(B^p)$ be nonnegative and the block Jacobi matrix $B$ be generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ with $|\zeta_L| = p - q$. Then the domain of convergence of the corresponding SOR method in the $(\rho(B), \omega)$-plane is*

$$(3.6) \qquad \Omega_1 = \{(\beta, \omega) : \beta \in [0, 1), \quad 0 < \omega < \omega_1(\beta) = \frac{2}{1+\beta}\}$$

*for $q$ even and*

$$(3.7) \qquad \Omega_2 = \{(\beta, \omega) : \beta \in [0, 1), \quad 0 < \omega < \omega_2(\beta)\}$$

*for $q$ odd, where $\omega_2(\beta)$ is a curve in the $(\beta, \omega)$-plane ($\beta = r_{max}$), is the upper bound of the domain of convergence, and can be given from the relationships (3.4) and (3.5) numerically.*

PROOF: For $q$ even we necessarily have $p$ odd. So, we are in case (ii) in which we have $\rho(B^p) = r_{min}^p$ or $\rho(B) = r_{min}$. For $\omega > 1$ the value of $r_{min}$ corresponds to an angle $\theta = p \cdot \frac{q}{p}\pi = q\pi$ for $\eta = 1$ and is given from the first equation of (2.10). Hence $\beta = (2 - \omega)/\omega$ or

$$(3.8) \qquad \omega = 2/(1 + \beta) =: \omega_1(\beta).$$

For $\omega < 1$, $r_{min} = 1$ and $\theta = 0$, therefore from the first equation of (2.10) we have that

$$(3.9) \qquad \beta = 1.$$

From (3.8), (3.9) and the analysis done we obtain that the convergence domain is the one defined in (3.6) and illustrated in Figure 3.1. For $q$ odd we are in either case (i) or (iii). In both of these cases we have $\rho(B) = r_{max}$ for $\omega > 1$. The value $r_{max}$ can be given from (3.4) and (3.5) as has already been mentioned. For $\omega < 1$, $r_{min} = 1$ and we also have $\beta = 1$. For each $\omega \in (0, 2)$ we obtain a unique value $\beta$. Obviously, all the pairs $(\beta, \omega)$ give the boundary curve $\omega_2(\beta)$. □

We note that the domains $\Omega_1$ and $\Omega_2$ do not coincide and neither of them is a subset of the other.

THEOREM 3.2. *Under the assumptions of Theorem 3.1, with the only exception being that $\sigma(B^p)$ is nonpositive, the corresponding domains of convergence are*

$$\Omega_3 = \{(\beta, \omega) : \omega \in (0, 2), \quad 0 \leq \beta < \beta_1(\omega)\} \tag{3.10}$$

*for $q$ even, and*

$$\Omega_4 = \{(\beta, \omega) : \omega \in (0, 2), \quad 0 \leq \beta < \beta_2(\omega)\} \tag{3.11}$$

*for $q$ odd. $\beta_1(\omega)$ is a curve in the $(\beta, \omega)$-plane ($\beta = r_{max}$) which constitutes the upper bound of $\Omega_3$ and can be found from (3.4) and (3.5) numerically. The curve $\beta_2(\omega)$ is given by $\beta = \frac{2-\omega}{\omega}$ for $\omega > 1$ and can be given from (3.4) and (3.5) for $\omega < 1$. Moreover, $\lim_{\omega \to 0^+} \beta_1(\omega) = \lim_{\omega \to 0^+} \beta_2(\omega) = 1/\cos\frac{\pi}{p}$.*

PROOF: For $q$ even we have $\rho(B^p) = r_{\max}^p$ for both cases $\omega > 1$ and $\omega < 1$, as in Theorem 3.1, and the domain of convergence is $\Omega_3$ given in (3.10). For $q$ odd, we are in cases (i) and (iii). Since $\sigma(B^p)$ is nonpositive, the boundary of this spectrum must be given from the smallest, in absolute value, polar radius of the boundary curve corresponding to the semiaxis with polar angle $\frac{\pi}{p}$. It is clear that this polar radius is $r_{\max}$ if $r_{\min}$ corresponds to the positive semiaxis and vice versa. So, from the cases (i) and (iii) it becomes clear that the polar radius in question is $r_{\min}$ for $\omega > 1$ and $r_{\max}$ for $\omega < 1$. In other words, $\beta_2(\omega) = \frac{2-\omega}{\omega}$ for $\omega > 1$ (or equivalently for $\beta < 1$) and $\beta_2(\omega)$ is found from (3.4) and (3.5) otherwise. It is an immediate conclusion that $\beta_1(2) = \beta_2(2) = 0$ and $\beta_1(1) = \beta_2(1) = 1$. To find the point of intersection of either of the two curves $\beta_1(\omega)$ and $\beta_2(\omega)$ with the axis $\omega = 0$ we have to follow an analysis analogous to the one in the application of stretched hypocycloids in [22]. For $\omega \to 0^+$ the right boundary of the region of convergence tends to that of a regular polygon as in [22]. Therefore $\beta_{\omega \to 0^+}(\omega) = \beta_{\omega \to 0^+}(\omega) \to 1/\cos\frac{\pi}{p}$ which completes the proof. □

The domains $\Omega_3$ and $\Omega_4$ for $(p, q) = (5, 2)$ and $(5, 3)$, respectively, are illustrated in Figures 3.2a and 3.2b.

## 4 Special cases and numerical examples.

### *4.1 Special cases.*

The special cases $(p, q) = (3, 1)$, $(4, 1)$ and $(5, 1)$ as well as $(p, q) = (3, 2)$, $(4, 3)$ and $(5, 4)$ have been studied by many researchers as was mentioned in the introductory Section 1. In the present section we study the special cases $(p, q) = (5, 2)$ and $(5, 3)$.

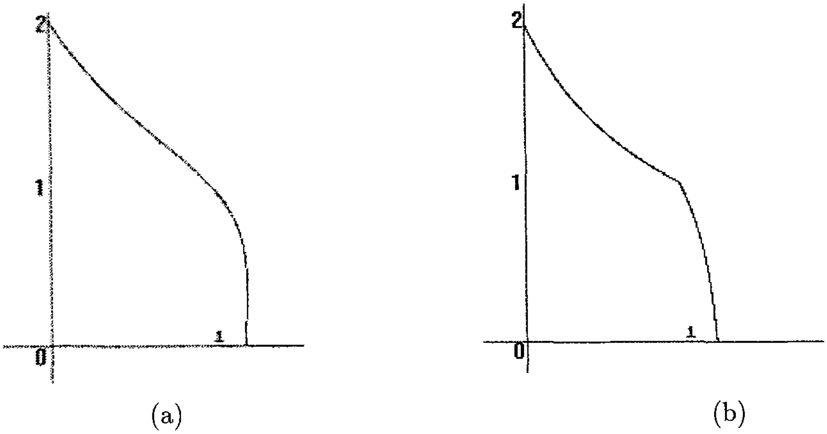



Figure 3.2: Convergence domains $\Omega_3$ and $\Omega_4$.

(i) $(\mathbf{p}, \mathbf{q}) = (\mathbf{5}, \mathbf{2})$: In this case equation (3.5) becomes

$$U_1(t) + (1-\omega)\eta U_2(t) = 0 \tag{4.1}$$

or

$$4(1-\omega)\eta t^2 + 2t - (1-\omega)\eta = 0. \tag{4.2}$$

The roots of (4.2) are

$$t_{+,-} = \frac{-1 \pm \sqrt{1+4(1-\omega)^2\eta^2}}{4(1-\omega)\eta} \tag{4.3}$$

In view of (4.3), relationship (3.5) becomes

$$x_{+,-} = \frac{1}{\omega\eta^{5/2}}\left(T_2(t_{+,-}) - (1-\omega)\eta T_3(t_{+,-})\right), \tag{4.4}$$

or

$$x_{+,-} = \frac{1}{\omega\eta^{5/2}}\left(2t_{+,-}^2 - 1 - (1-\omega)\eta(4t_{+,-}^3 - 3t_{+,-})\right), \tag{4.5}$$

or

$$x_{+,-} = \frac{(1-(1-\omega)^2\eta^2)(1 \mp \sqrt{1+4(1-\omega)^2\eta^2})}{2\omega(1-\omega)^2\eta^{12/5}}. \tag{4.6}$$

The smallest in absolute value of $x_+$ and $x_-$ gives the value of $r_{\max}$. It is obvious from (4.6) that $x_+ < 0$, hence $r_{\max} = |x_+|$. The value of $r_{\min}$ is given from (3.2).

Therefore

$$r_{\min} = \frac{1+(1-\omega)\eta}{\omega\eta^{5/2}}. \tag{4.7}$$

Obviously the general theory developed in the previous sections is confirmed in this specific case, where, in addition, we have an analytical expression for $r_{\max}$.

So, if $\sigma(B^5)$ is nonnegative, by substituting 1 for $\eta$ we obtain the convergence domain $\Omega_1$ as this is given in Theorem 3.1. On the other hand, if $\sigma(B^5)$ is nonpositive we obtain the convergence domain $\Omega_3$ from Theorem 3.2. In the present case $\beta_1(\omega)$ is given by

$$\beta_1(\omega) = |r_{\max}| = \frac{2(2-\omega)}{1+\sqrt{1+4(1-\omega)^2}}. \tag{4.8}$$

Differentiating $\beta_1(\omega)$ wrt $\omega$ it is readily proved that $\beta_1(\omega)$ is a strictly increasing function in $(0,1/3]$ and a strictly decreasing one in $[1/3,1)\bigcup\{1\}\bigcup(1,2)$. So,

$$\max_{\omega}\beta_1(\omega) = \beta(\frac{1}{3}) = \frac{5}{4}.$$

For $\omega$ tending to zero we have from (4.8) that

$$\lim_{\omega\to 0^+}\beta_1(\omega) = \beta_1(0) = \frac{4}{1+\sqrt{5}} = \frac{1}{\cos\frac{\pi}{5}} \tag{4.9}$$

which confirms the corresponding part of Theorem 3.2.

(ii) $(\mathbf{p},\mathbf{q}) = (\mathbf{5},\mathbf{3})$: Equation (3.5) becomes now

$$U_2(t) + (1-\omega)\eta U_1(t) = 0 \tag{4.10}$$

or

$$4t^2 + 2(1-\omega)\eta t - 1 = 0 \tag{4.11}$$

with roots

$$t_{+,-} = \frac{-(1-\omega)\eta \pm \sqrt{4+(1-\omega)^2\eta^2}}{4}. \tag{4.12}$$

As in the previous case, relation (3.5) gives

$$x_{+,-} = \frac{[1-(1-\omega)^2\eta^2][(1-\omega)\eta \mp \sqrt{4+(1-\omega)^2\eta^2}]}{2\omega\eta^{3/5}}. \tag{4.13}$$

The value of $r_{\max}$ this time becomes

$$r_{\max} = \begin{cases} |x_+|, & \text{if} \quad \omega \le 1 \\ |x_-| & \text{if} \quad \omega \ge 1 \end{cases} \tag{4.14}$$

while

$$r_{\min} = \frac{1+(1-\omega)\eta}{\omega\eta^{5/3}} \tag{4.15}$$

For the nonnegative case the convergence domain $\Omega_2$ is obtained from Theorem 3.1. The boundary curve $\omega_2(\beta)$ is then given by

$$\beta(\omega) = |r_{\max}| = \frac{1}{2}(2-\omega)(1-\omega+\sqrt{4+(1-\omega)^2}). \tag{4.16}$$

For the nonpositive case the convergence domain is now $\Omega_4$ given in Theorem 3.2, and $\beta_2(\omega)$ is given by the function

$$\beta_2(\omega) = \begin{cases} \frac{1}{2}(2-\omega)(\omega-1+\sqrt{4+(1-\omega)^2}), & \text{if } \omega \le 1 \\ \frac{2-\omega}{\omega}, & \text{if } \omega \ge 1. \end{cases} \tag{4.17}$$

It is readily proved that the functions in (4.16), (4.17) are strictly decreasing ones. Also it is obtained that

$$\lim_{\omega \to 0^+} \beta_2(\omega) = \beta_2(0) = \sqrt{5} - 1 = \frac{1}{\cos\frac{\pi}{5}}, \tag{4.18}$$

confirming the corresponding part of the theory.

*4.2 Numerical examples.*

For the verification of the theoretical findings of previous works as well as of the present one we do not consider nonsingular cyclic linear systems since the theory then can be trivially verified. Instead we work out a number of numerical examples from Markov chain analysis. Although the cyclic SOR theory for singular systems is at the early stages of its development we consider it a challenge to work out these examples in order to investigate whether and when the theory of the cyclic SOR for nonsingular systems carries over to singular ones. Suppose then that

$$x = Bx \tag{4.19}$$

is the singular matrix equation that gives the stationary (probability) distribution vector $x \in \mathbb{R}^{15}$, $||x||_1 = 1$, with $B \in \mathbb{R}^{15,15}$ (and index$(I - B) = 1$) the stochastic transition probability matrix. Suppose also that $B = B_j$, $j = 1(1)4$, is $5 \times 5$ block weakly cyclic of index 5 generated, in turn, by the cyclic permutations $\sigma^{(1)} = (5,4,3,2,1)$, $\sigma^{(2)} = (5,3,1,4,2)$, $\sigma^{(3)} = (5,2,4,1,3)$, $\sigma^{(4)} = (5,1,2,3,4)$, corresponding to the pairs $(p,q) = (5,1)$, $(5,2)$, $(5,3)$, $(5,4)$, respectively. In all the cases below we take the nonidentically zero blocks of $B$ to be equal to

$$\widetilde{B} = \begin{bmatrix} 0.90 & 0.05 & 0.05 \\ 0.05 & 0.90 & 0.05 \\ 0.05 & 0.05 & 0.90 \end{bmatrix}.$$

For example

$$B = B_2 = \begin{bmatrix} 0 & 0 & 0 & \widetilde{B} & 0 \\ 0 & 0 & 0 & 0 & \widetilde{B} \\ \widetilde{B} & 0 & 0 & 0 & 0 \\ 0 & \widetilde{B} & 0 & 0 & 0 \\ 0 & 0 & \widetilde{B} & 0 & 0 \end{bmatrix}.$$

Table 4.1: Numerical results

| $(p,q)$ | $\omega$ | Number of Iterations | $(p,q)$ | $\omega$ | Number of Iterations |
|---|---|---|---|---|---|
| | $\widehat{\omega}_1 - 0.05$ | 21 | | .95 | 42 |
| $(5,1)$ | $\widehat{\omega}_1$ | 13 | $(5,2)$ | $\widehat{\omega}_2 = 1$ | 37 |
| | $\widehat{\omega}_1 + 0.05$ | 16 | | 1.05 | 93 |
| | 0.95 | 58 | | 0.95 | 73 |
| $(5,3)$ | $\widehat{\omega}_3 = 1$ | 53 | $(5,4)$ | $\widehat{\omega}_4 = 1$ | 70 |
| | 1.05 | 96 | | 1.05 | 165 |

The eigenvalues of $\widetilde{B}$ are readily found to be 1, 0.85, 0.85 and therefore those of $B^5 = B_j^5$, $j = 1(1)4$, are the numbers 1 of multiplicity five and $0.85^5$ of multiplicity ten. So the eigenvalues of $B$ are the numbers $e^{i\frac{2\pi k}{5}}$, $k = 0(1)4$, of multiplicity one and $0.85e^{i\frac{2\pi k}{5}}$, $k = 0(1)4$, of multiplicity two each, hence $\text{index}(I - B) = 1$. Also it can be proved in a way analogous to that in Theorem 3.1 of [9] that $\text{index}(I - \mathcal{L}_\omega) = 1$, $\forall\, \omega \in \mathbb{C}\backslash\{0, \frac{p}{p-q}\}$. Therefore by virtue of the theory in [2] [9], [14], [13] one can use $\sigma(B)\backslash\left\{e^{i\frac{2\pi k}{5}},\ k = 0(1)4\right\}$ in the place of $\sigma(B)$ and derive the optimal SOR parameters using the results known from the nonsingular case. So, instead of considering (4.19) and therefore the block Jacobi iterative scheme

$$x^{(m+1)} = Bx^{(m)},\ m = 0, 1, 2, \ldots, \tag{4.20}$$

where $x^{(0)} \in \mathbb{R}^{15}$, $||x^{(0)}||_1 = 1$, is the initial stationary (probability) distribution vector, we consider

$$x^{(m+1)} = \mathcal{L}_\omega x^{(m)},\ m = 0, 1, 2, \ldots, \tag{4.21}$$

with the same initial vector $x^{(0)}$.

Since the spectra $\sigma(B_j^5)$, $j = 1(1)4$, are nonnegative the corresponding optimal SOR parameter, for both the nonsingular and singular consistently ordered cases, will be given as the unique real root in $(1, \frac{5}{4})$ of the equation

$$(0.85\omega_1)^5 - \frac{5^5}{4^4}(\omega_1 - 1) = 0 \tag{4.22}$$

which is $\widehat{\omega}_1 = 1.045379$, (see, e.g., [26, 27, 31, 9, 14]), while for the nonsingular inconsistently ordered cases it will be $\widehat{\omega}_j = 1$, $j = 2, 3, 4$ (see, e.g., [28], [31]). Regions of convergence for the corresponding nonsingular cases have been studied and determined in [10] for $(p,q) = (5,1)$, in [11] for $(p,q) = (5,4)$ and in the present work for the other two cases.

In the numerical examples below we investigate experimentally whether the theoretical results that hold in the various nonsingular cases can hold in the corresponding singular ones. More specifically:

i) The optimal SOR parameter $\omega$ of the nonsingular case is it also the optimal one for the corresponding singular case? (This was already proved to hold true in the case of a consistently ordered matrix $B$, in our case for $(p,q) = (5,1)$, in [9].)

ii) An $\omega$ close to the optimal one and chosen from the region of convergence that has been determined for the nonsingular case does it also belong to the convergence region of the singular case? (Continuity arguments can certainly apply to values of $\omega$ very close to the optimal one and guarantee convergence in the singular case too.)

iii) Is it overestimation or underestimation of the optimal parameter that gives better results in the singular case? (This question has been extensively studied theoretically for nonnegative and nonpositive spectra and for both the nonsingular and singular cases but only for $p = 2$ (see, e.g., [27], [13]).)

In our experiments for the four cases considered we have chosen various values of $\omega$ around the optimal one. In the self-explained accompanying Table 4.1 only the results (number of iterations) obtained for

$$\omega = \widehat{\omega}_j - 0.05,\ \widehat{\omega}_j,\ \widehat{\omega}_j - 0.05, \quad j = 1(1)4,$$

are illustrated. In these examples we have taken $x^{(0)} = [y_1^T, y_2^T, \ldots, y_5^T]^T$, where $y_j^T = [0.2, 0, 0]$, $j = 1(1)5$, and as a criterion to stop the iterations

$$\frac{||x^{(m)} - x^{(m-1)}||_\infty}{||x^{(m)}||_\infty} \leq 0.5 \times 10^{-6}$$

while all the calculations have been carried out in double precision arithmetic. The results, as shown in Table 1, seem to agree almost completely with the ones that the theory known for the nonsingular and singular cases predicts. Specifically: In all the cases considered the optimal $\omega's$ chosen are also optimal for the corresponding singular cases. Also, for the consistently ordered case one obtains the best optimal results which become worse and worse as the basic matrix $B$ moves away from the consistently ordered case, something which was expected. Finally, in the consistently ordered case an overestimation of $\widehat{\omega}$ gives better results than an underestimation of it and this is in agreement with what is known for the case $p = 2$. In all the other (inconsistently ordered) cases examined it seems that an underestimation of $\widehat{\omega}$ gives better results than an overestimation of it. This observation is in agreement with what is known for the consistently ordered nonpositive case where $\widehat{\omega} < 1$ rather than with that of the nonnegative one. A possible explanation for this may be the following. The fact that the modulus of the second largest eigenvalue of $B$ is relatively large has as a consequence that the value of the chosen $\omega$ that is greater than one is very close to the upper boundary curve of the SOR convergence region (e.g., for $(p,q) = (5,2)$ the upper bound of $\omega$ in the nonsingular case is $\frac{2}{1+0.85} = 1.081081$). As a result of this the convergence of the (singular) SOR method becomes very slow.

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