On the Optimum Relaxation Factor Associated With *p*-Cyclic Matrices

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ABSTRACT

Assume that the matrix coefficient of the nonsingular linear system $Ax = b$ belongs to the class of the generalized consistently ordered $(p-q, q)$ matrices, where p and q are relatively prime. It is well known that under the additional assumption that the pth powers of the eigenvalues of the Jacobi matrix T associated with A are nonnegative (nonpositive), the problem of determining the optimum relaxation factor that maximizes the asymptotic convergence rate of the successive overrelaxation method for the solution of $Ax = b$ has been solved in many cases. Thus, in the works by Young, by Varga, and by Nichols and Fox, the problem has been solved in the nonnegative case for any (p, q). In the nonpositive case, in view of the work by Kredell, by Niethammer, de Pillis, and Varga, by Galanis, Hadjidimos, and Noutsos, and by Wild and Niethammer, the corresponding problem seems to be more difficult; it has been solved only for $q = p - 1$. The present work is a contribution towards the solution of the problem in the latter case. In particular, we study the case $q = 1$, $p \geqslant 3$, with detailed results for $p = 3, 4$.

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1. INTRODUCTION AND PRELIMINARIES

Consider the nonsingular linear system

$$Ax = (D - L - U)x = b, \tag{1.1}$$

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where A is partitioned into an $n \times n$ block form with A_{ii}, $i = 1(1)n$, square and nonsingular, and D, L, and U are respectively block diagonal, block strictly lower triangular, and block strictly upper triangular relative to the partitioning considered. Suppose that the eigenvalues of the matrix

$$D^{-1}(\alpha L + \alpha^{-\Theta} U), \qquad \Theta = \frac{q}{p-q}, \quad \alpha \neq 0, \tag{1.2}$$

where p and q $(p > q > 0)$ are relatively prime integers, are fixed no matter what value for $\alpha \neq 0$ is chosen. When $\Theta = 1$, i.e., $p = 2$, $q = 1$, the matrix A has been defined by Young [11, 12] to be consistently ordered. More generally, when $q = p - 1$, we have the p-cyclic consistently ordered matrices defined by Varga [7, 8]. Verner and Bernal [9] (see also [4]) considered cases which were more general, with q not necessarily equal to $p - 1$. So we have the *generalized consistently ordered* $(p - q, q)$ matrices [or $(p - q, q)$ GCO matrices]. In this paper, in particular, we study the case $q = 1$, $p \geqslant 3$, with detailed results for $p = 3, 4$.

Let A in (1.1) be a $(p - q, q)$ GCO matrix. Then the eigenvalues of the successive overrelaxation (SOR) matrix $\mathscr{L}_\omega$ and of the Jacobi matrix T associated with A,

$$\mathscr{L}_\omega := (D - \omega L)^{-1}[(1 - \omega)D + \omega U] \tag{1.3}$$

and

$$T := D^{-1}(L + U) \tag{1.4}$$

respectively, are connected through the relationship

$$(\lambda + \omega - 1)^p = \mu^p \omega^p \lambda^q, \tag{1.5}$$

where $\lambda \in \sigma(\mathscr{L}_\omega)$, $\mu\theta^j \in \sigma(T)$, $j = 0(1)p - 1$, and $\theta = e^{2\pi i/p}$ (see [9] or [4]), with $\sigma(\cdot)$ denoting the spectrum of a matrix. The relationship (1.5) is due to Verner and Bernal [9] and generalizes the famous equations of Young [11], $(p, q) = (2, 1)$, and of Varga [7], $(p, q) = (p, p - 1)$, $p \geqslant 3$.

The determination of the optimum relaxation factor $\omega(\omega_{\text{opt}})$ so that the asymptotic convergence rate of the SOR method for the solution of (1.1) is maximized [or equivalently $\rho(\mathscr{L}_\omega)$ is minimized, where $\rho(\cdot)$ denotes the spectral radius of a matrix] has attracted the interest of many researchers. So, in several cases of practical and theoretical interest ω_{opt} has been determined. In particular, for $\sigma(T^p)$ nonnegative, ω_{opt} was determined by Young [11], $(p, q) = (2, 1)$; by Varga [7], $(p, q) = (p, p - 1)$, $p \geqslant 3$; and by Nichols and Fox [4], (p, q), $p \geqslant 3$, $q \leqslant p - 2$. For $\sigma(T^p)$ nonpositive the very first ω_{opt} was determined by Kredell [3], $(p, q) = (2, 1)$. Rather recently Niethammer, de Pillis, and Varga [5], motivated by a least-squares problem [1, 6], determined ω_{opt} for $(p, q) = (3, 2)$, and very recently Galanis, Hadjidimos, and Noutsos [2] and independently Wild and Niethammer [10] determined it for $(p, q) = (p, p-1)$, $p \geqslant 4$. To the best of our knowledge nothing has been done in the case of $\sigma(T^p)$ nonpositive for $p \geqslant 3$, $q \leqslant p-2$ analogous to the work Nichols and Fox [4] did in the nonnegative case.

To start a discussion and contribute towards the solution of the problem arising in the latter case, we have begun a study of the case $q = 1$, $p \geqslant 3$, which constitutes, in a sense, the complement of the $q = p - 1$ one. As the reader will find out, matters do not appear to be as straightforward as one would expect them to be, having in mind the analysis in the general case of $\sigma(T^p)$ nonnegative [4]. For example: In Section 2 it is shown that if there exist values of ω for which the SOR method converges, then $\omega_{opt} \in (0, 1]$. This is something which would be expected. However, if $\omega_{opt} \neq 1$ (contrary to what is known for the corresponding nonnegative case, where $\omega_{opt} = 1$ [4])—and this is indeed the case at least for $p = 3$ and also for a major subcase of $p = 4$, as is shown in Section 3—then $\lambda(\omega_{opt})$ in (1.5) does not correspond to a double real zero as it does in the cases of nonnegative and nonpositive $\sigma(T^p)$ for $q = p - 1$. It corresponds to a pair of complex conjugate zeros. We would also like to point out that some of the results of Section 2 hold more generally than for the $q = 1$ case treated there, but it is not known as yet if they can cover the entire class of pairs (p, q), $p \geqslant 3$, $q \leqslant p - 2$. Finally, the main theorem, which is proved in Section 3 and covers the cases $p = 3, 4$, is stated as follows:

THEOREM 1. *Let A be partitioned in blocks A_{ij}, $i, j = 1(1)n$, where A_{ii}, $i = 1(1)n$, are square and nonsingular. Let $D = \operatorname{diag}(A_{11}, \dots, A_{nn})$, and A be written as in* (1.1). *Assume that, relative to its partitioning, A is*

(I) (2, 1) GCO *or*
(II) (3, 1) GCO,

and let $\mathscr{L}_\omega$ in (1.3) *and T in* (1.4) *denote the block SOR and block Jacobi matrices associated with A respectively, and $\beta := \rho(T)$. Then:*

In case (I), *if $\sigma(T^3)$ is nonpositive, then:*

(i) *For $\beta < 2$ there exists a value ω_{opt} for ω, the unique positive real root of the equation*

$$(1+\omega)^2\beta^3 - 8(1-\omega) = 0, \tag{1.6a}$$

viz.

$$\omega_{opt} = \frac{-(4+\beta^3) + 4(1+\beta^3)^{1/2}}{\beta^3}, \tag{1.6b}$$

in (0, 1), *such that for all* $\omega \neq \omega_{opt}$,

$$\rho(\mathscr{L}_\omega) > \min_\omega \rho(\mathscr{L}_\omega) = \left(1 - \omega_{\text{opt}}^2\right)^{1/2}. \tag{1.7}$$

(ii) *For* $\beta \geqslant 2$ *there holds*

$$\rho(\mathscr{L}_\omega) \geqslant 1. \tag{1.8}$$

In case (II), *if* $\sigma(T^4)$ *is nonpositive, then*:

(i) *For* $0 < \beta \leqslant 1/\sqrt[4]{8}$

$$\omega_{\text{opt}} = 1, \tag{1.9}$$

and for all $\omega \neq \omega_{opt}$

$$\rho(\mathscr{L}_\omega) \geqslant \min_\omega \rho(\mathscr{L}_\omega) = \beta^{4/3}. \tag{1.10}$$

(ii) *For* $1/\sqrt[4]{8} < \beta < \sqrt{2}$ *there exists a value* ω_{opt} *for* ω, *the unique positive real root of the equation*

$$\omega^2 r^3 \beta^2 - \left[r^2 - (1 - \omega)\right]^2 \left[r^2 + (1 - \omega)\right] = 0, \tag{1.11}$$

with

$$r = \left(\frac{\omega + \left(16 - 8\omega - 7\omega^2\right)^{1/2}}{4}\right)^{1/2}, \tag{1.12}$$

in (0, 1), *such that for all* $\omega \neq \omega_{opt}$

$$\rho(\mathscr{L}_\omega) > \min_\omega \rho(\mathscr{L}_\omega) = \frac{\omega_{\text{opt}} + \left(16 - 8\omega_{\text{opt}} - 7\omega_{\text{opt}}^2\right)^{1/2}}{4}. \tag{1.13}$$

(iii) *For* $\beta \geqslant \sqrt{2}$ *there holds*

$$\rho(\mathscr{L}_\omega) \geqslant 1.$$

NOTE. The trivial case $\rho(T) = 0$ is not considered in Theorem 1 or in the general case $(p, q) = (p, 1)$, for in such a case it can readily be seen from (1.5) that $\omega_{\text{opt}} = 1$ and $\rho(\mathcal{L}_{\omega_{\text{opt}}}) = 0$.

2. ANALYSIS OF THE GENERAL CASE $(p, q) = (p, 1)$

We begin our analysis with Equation (1.5). Since $\mu^p \leqslant 0$, where μ is any eigenvalue of the Jacobi matrix T in (1.4), we set $\mu^p = -\nu^p$, with $\nu \in (0, \beta := \rho(T)]$ fixed, and extract pth roots ($q = 1$) to obtain $\lambda + \omega - 1 = \nu\omega\lambda^{1/p}\exp[i(2k+1)\pi/p]$, where $\lambda^{1/p}$ is any pth root of λ and k is any integer. Putting $z := \lambda^{1/p}\exp[i(2k+1)\pi/p]$, we have the equivalent equation

$$g(z, \omega) := z^p + \omega\nu z + 1 - \omega = 0. \tag{2.1}$$

Let $z_j := z_j(\omega)$, $j = 1(1)p$, denote the zeros of (2.1). Since our objective is to minimize $\rho(\mathcal{L}_\omega)$ as a function of ω and $\lambda = -z^p$, we try, equivalently, to minimize $\max_j |z_j|$, for a fixed $\nu \in (0, \beta]$, as a function of $\omega \in (0, 2)$; for if $\omega \notin (0, 2)$ then $\rho(\mathcal{L}_\omega) \geqslant 1$. Then we consider the largest possible value of the minimum in question over all $\nu \in (0, \beta]$. (Note: The trivial case $\nu = 0$ is not examined here or in Section 3, since it can be considered as a limiting case and can be covered by the analysis that follows by using continuity arguments.) First we prove that for a given ν the aforementioned minimum cannot take place for any $\omega \in (1, 2)$. For this we have:

PROPOSITION 1. *For any* $\omega \in (1, 2)$ *Equation* (2.1) *has always at least one zero with modulus strictly greater than* $\max_j |z_j(1)| = \nu^{1/(p-1)}$.

Proof. By Descartes's rule of signs it is readily checked that for p odd (2.1) has precisely one real zero, which is positive, while for p even it has precisely two real zeros, one negative and one positive. If we put $y = z^p$, then from (2.1) we take

$$y + \omega\nu z + 1 - \omega = 0. \tag{2.2}$$

From (2.2) we see that if $\operatorname{Re} z \leqslant 0$ then $\operatorname{Re} y > 0$, while if $\operatorname{Im} z \gtreqless 0$ then $\operatorname{Im} y \lesseqgtr 0$. We follow Nichols and Fox [4] and differentiate (2.2) and $y = z^p$

with respect to ω. After eliminating ν by using (2.1) we obtain

$$\frac{\partial y}{\partial \omega} = \frac{py(1+y)}{\omega[(p-1)y+\omega-1]}, \tag{2.3}$$

and from this

$$\operatorname{Re}\frac{\partial y}{\partial \omega} = \frac{p\{R(R+1)[(p-1)R+\omega-1] + [(p-1)(R+1)-(\omega-1)]I^2\}}{D}, \tag{2.4}$$

$$\operatorname{Im}\frac{\partial y}{\partial \omega} = \frac{p[(p-1)R^2+(2R+1)(\omega-1)+(p-1)I^2]I}{D},$$

where we have set

$$R := \operatorname{Re} y, \quad I := \operatorname{Im} y, \quad D := \omega\{[(p-1)R+(\omega-1)]^2+(p-1)^2I^2\}. \tag{2.5}$$

From (2.4)–(2.5) it is readily concluded that

$$R \geqslant 0 \quad \text{implies} \quad \operatorname{Re}\frac{\partial y}{\partial \omega} > 0,$$

$$R \geqslant 0 \text{ and } I \gtreqless 0 \quad \text{implies} \quad \operatorname{Im}\frac{\partial Y}{\partial \omega} \gtreqless 0. \tag{2.6}$$

Obviously, at $\omega = 1$ and for $p \geqslant 4$ (the proof for $p = 3$ will be given in Section 3), (2.1) has at least one zero z with $\operatorname{Re} z < 0$, $\operatorname{Im} z \geqslant 0$ and for which $|z(1)| = \nu^{1/(p-1)}$. This particular zero we are considering will have for all $\omega \in (1,2)$ either $\operatorname{Re} z < 0$ and $\operatorname{Im} z = 0$ or $\operatorname{Re} z < 0$ and $\operatorname{Im} z > 0$. It is clear that in the first case we are referring to the real negative zero of (2.1) for even p ($\geqslant 4$), and in the second case to one of the zeros in the second quadrant for odd p ($\geqslant 5$). It is also evident that in the latter case $\operatorname{Re} z$ cannot become 0 for any $\omega \in (1,2)$, because then z^p will also be purely imaginary, leading to a contradiction, for from (2.1) $\omega = 1$. Moreover, $\operatorname{Im} z$ cannot become 0 for any $\omega \in (1,2)$, for then the zero in question and its complex conjugate will become a *double* real negative zero for (2.1), which is not possible. Based on the previous analysis and on the conclusions (2.6), we have that the image of the corresponding y in the complex plane will have a strictly increasing real part $(R > 0)$ and a nondecreasing imaginary part $(I \geqslant 0)$ as ω increases from 1 to 2. This implies that the modulus of y increases with respect to ω and so does the modulus of z, which concludes the proof of the proposition. ■

As a corollary to Proposition 1 we have that:

PROPOSITION 2. *The minimum of* $\rho(\mathscr{L}_\omega)$ *will take place for some* $\omega \in (0, 1]$.

In analogy with what is known, the result just obtained would be expected. This is because for $\sigma(T^p)$ *nonpositive*, with $(p, q) = (p, p-1)$, one has

$$\omega_{\text{opt}} \in \left(\frac{p-2}{p-1}, 1 \right)$$

(see [3], [6], [2], and [10]). Also, for nonnegative $\sigma(T^p)$, with $(p, q) = (p, 1)$—a special case of that treated in [4]—one has $\omega_{\text{opt}} = 1$. However, what is stated and proved in the sequel, which applies at least in the cases $p = 3$ and 4 we are examining in the next section, is contrary to what is known from similar cases so far.

PROPOSITION 3. *Let* $\omega_{\text{opt}} \neq 1$. *Then* $\max_j |z_j(\omega_{\text{opt}})|$, *where* z_j *are the zeros of* (2.1) [*or equivalently max* $|\lambda(\omega_{\text{opt}})|$ *of* (1.5) *with* $q = 1$] *corresponds to a pair of complex conjugate zeros of* (2.1) [*or equivalently of* (1.5)] *and not to a double real zero.*

Proof. Applying Descartes's rule of signs for $\omega \in (0, 1)$, it can be seen that for p odd $g(z, \omega)$ has precisely one real (negative) zero, while for p even it has either two real (negative) or no real zeros. For p odd let z_p be the real (negative) zero of (2.1) and $(z_1, z_2), (z_3, z_4), \dots, (z_{p-2}, z_{p-1})$ the pairs of complex conjugate zeros. At $\omega = 1$ one has $|z_j| > |z_p| = 0$, $j = 1(1)p-1$. So, if our assertion were not true, there would be an $\omega \in (0, 1)$ at which $|z_p| \geqslant |z_j|$, $j = 1(1)p-1$. Recalling that $\Pi_{j=1}^{p} z_j = \omega - 1$, the previous inequality would give $-z_p^p \geqslant 1 - \omega$, or $z^p + 1 - \omega \leqslant 0$. However, (2.1) implies that $\omega \nu z_p \geqslant 0$ or, equivalently, $z_p \geqslant 0$, which contradicts the fact that z_p is negative for $\omega \in (0, 1)$. For p even we observe that $g(z, 0) = z^p + 1$ has all its zeros complex while $g(z, 1) = z^p + \nu z$ has 0 and $-\nu^{1/(p-1)}$ as its two real zeros. Using again the substituting $y = z^p$ as in Proposition 1 for the two real roots, we can find out from (2.3) that as ω decreases from the value 1, the largest $y > 0$ strictly decreases while the smallest $y > 0$ strictly increases until they become equal for $\omega = \omega_c \in (0, 1)$. The value ω_c is the unique positive real zero in (0, 1) of the equation

$$f(\omega) := (\omega\nu)^p - p^p(p-1)^{1-p}(1-\omega)^{p-1}, \tag{2.7}$$

and the double value of y (or of z) is given by $(1-\omega_c)/(p-1)$ (or by $-[(1-\omega_c)/(p-1)]^{1/p}$). That at $\omega = \omega_c$ the double value of the zero

$$z = -\left(\frac{1-\omega_c}{p-1}\right)^{1/p}$$

cannot lead to an optimum ω is proved as follows. We have $\Pi_{j=1}^{p}(z_j) = 1 - \omega_c$ at $\omega = \omega_c$; therefore $(\max |z_j(\omega_c)|)^p \geqslant 1 - \omega_c$. Substituting in the left hand side the value for the double z found before, we have $(1-\omega_c)/(p-1) \geqslant 1 - \omega_c$. This leads to the contradiction $2 \geqslant p$, which concludes the proof. ∎

NOTE. Before we close this section we would like to clarify a point in connection with the value of ω_{opt} in the proof of Proposition 3 in case p is even. For this, let $\omega_d \in (0, 1)$ be the value of ω at which $\max |z_j(\omega)|$, taken over all complex z_j's, is minimized, and let m_d be this minimum value. It is clear that if $\omega_d \in (0, \omega_c]$, with ω_c being defined in the proof of Proposition 3, then $\omega_{\text{opt}} = \omega_d$. However, if $\omega_d \in (\omega_c, 1)$, we distinguish two cases. So, if $|z_{p-1}(\omega_d)| \leqslant m_d$, where $z_{p-1}(\omega)$ is the largest in modulus of the two real negative zeros z_{p-1} and z_p of (2.1) for $\omega \in (\omega_c, 1)$, then $\omega_{\text{opt}} = \omega_d$. If, on the other hand, $|z_{p-1}(\omega_d)| > m_d$, let $\omega_e \in (\omega_c, \omega_d)$ denote the smallest value of ω at which $|z_{p-1}(\omega_e)| = \max_{j=1(1)p-2} |z_j(\omega_e)|$. As is obvious then, $\omega_{\text{opt}} = \omega_e$.

3. THE PROOF OF THEOREM 1.

(I) $p = 3$: Let z_1, z_2, and z_3 be the three zeros of (2.1), and let the first two be the complex conjugate ones. We have

$$\begin{aligned} z_1 + z_2 + z_3 &= 0, \\ (z_1 + z_2) z_3 + z_1 z_2 &= \omega\nu, \\ z_1 z_2 z_3 &= \omega - 1. \end{aligned} \tag{3.1}$$

Eliminating $z_1 + z_2$ and z_3 from (3.1), one obtains

$$r^3 - \omega \nu r^2 - (1 - \omega)^2 = 0, \tag{3.2}$$

where we have set $r = z_1 z_2 = |z_1(\omega)|^2$. Differentiating (3.2) with respect to ω, we take

$$\frac{\partial r}{\partial \omega} = \frac{r[r^3 - (1 - \omega^2)]}{\omega[r^3 + 2(1 - \omega)^2]}, \tag{3.3}$$

where ν was eliminated by using (3.2). Obviously for $\omega \in [1, 2)$ there is no value of $r > 0$ for which the derivative in (3.3) vanishes. In fact, we have always $\partial r / \partial \omega > 0$, showing that $|z_1(\omega)|$ strictly increases with ω in $[1, 2)$. This observation completes the proof of Proposition 1. On the other hand for any $\omega \in (0, 1)$, r of (3.2) assumes the minimum value $(1 - \omega^2)^{1/3} > (1 - \omega)^{2/3} > |z_3(\omega)|^2$. Substituting this value for r in (3.2), we obtain

$$h(\omega) := (1 + \omega)^2 \nu^3 - 8(1 - \omega) = 0. \tag{3.4}$$

Requiring a solution ω of (3.4) to be in (0, 1), we must have $h(0)h(1) < 0$, from which the sufficient condition $\nu < 2$ is produced. Since in addition $\partial h / \partial \omega > 0$, the value of ω ($= \omega_{\text{opt}}$) obtained in this way is unique and is given by

$$\omega_{\text{opt}} = \frac{-(4 + \nu^3) + 4(1 + \nu^3)^{1/2}}{\nu^3}. \tag{3.5}$$

The condition $\nu < 2$ is also a necessary one for the SOR method to converge, for if $\nu \geqslant 2$, the minimum value of r would be attained at $\omega = 0$, for which $\rho(\mathscr{L}_0) = 1$. One must bear in mind that the analysis in this section was made for any $\nu \in (0, \beta]$ fixed. So, in order to determine the overall optimum, one should determine the largest possible value for the minimum $r = (1 - \omega_{\text{opt}}^2)^{1/3}$ just obtained. Evidently ω_{opt} must be as small as possible. Differentiating then (3.4) with respect to ν, considering ω as a function of ν, we get

$$\frac{\partial \omega}{\partial \nu} = -\frac{3\nu^2(1 + \omega)^2}{2[\nu^3(1 + \omega) + 4]} < 0.$$

This effectively shows that ω decreases with ν increasing in $(0, \beta]$. Consequently, the optimum results are obtained for $\nu = \beta$. This concludes the analysis for the particular case $p = 3$.

(II) $p = 4$: Let $z_j := z_j(\omega)$ be the four zeros of (2.1). Since at least two of them will be complex, let them be z_1 and z_2. This time we will have

$$\begin{aligned} z_1 + z_2 + z_3 + z_4 &= 0, \\ z_1 z_2 + (z_1 + z_2)(z_3 + z_4) + z_3 z_4 &= 0, \\ z_1 z_2(z_3 + z_4) + z_3 z_4(z_1 + z_2) &= -\omega\nu, \\ z_1 z_2 z_3 z_4 &= 1 - \omega. \end{aligned} \tag{3.6}$$

Eliminating $z_1 + z_2$, $z_3 + z_4$, and $z_3 z_4$ from the equations of (3.6), setting $r = z_1 z_2 = |z_1(\omega)|^2$, and imposing the restriction

$$r \geqslant (1 - \omega)^{1/2} \tag{3.7}$$

to guarantee that at least when all z_j's, $j = 1(1)4$, are complex, z_1 and z_2 constitute the pair with the largest modulus, after some manipulation one obtains

$$[r^2 - (1 - \omega)]^2 [r^2 + (1 - \omega)] - \omega^2 r^3 \nu^2 = 0. \tag{3.8}$$

Differentiating (3.8) with respect to ω, solving for $\partial r / \partial \omega$, and substituting into the resulting equation ν^2 from (3.8), we finally have

$$\frac{\partial r}{\partial \omega} = \frac{r[2r^4 - \omega r^2 - (1 - \omega)(2 + \omega)]}{\omega[3r^4 + 2(1 - \omega)r^2 + 3(1 - \omega)^2]}. \tag{3.9}$$

It is readily seen from (3.9), having in mind the restriction (3.7), that r $[\geqslant (1 - \omega)^{1/2}]$ becomes a minimum if and only if

$$r = \left(\frac{\omega + (16 - 8\omega - 7\omega^2)^{1/2}}{4} \right)^{1/2}. \tag{3.10}$$

Since $\lim_{\omega \to 1^-} r = 1/\sqrt{2}$, a continuity argument implies that even for $\omega \in (\omega_c, 1)$ the pair z_1, z_2 corresponds to the product of the two complex conjugate zeros of (2.1) and not to the corresponding product of the real zeros z_3 and z_4, because $z_3(1) z_4(1) = 0$. To simplify matters we follow a slightly different analysis from the one in case $p = 3$. For this, assume that $\omega \in (0, 1]$ is fixed and r varies, so that $r \geqslant \{[\omega + (16 - 8\omega - 7\omega^2)^{1/2}]/4\}^{1/2}$, and r, ω satisfy (3.8). In this way r becomes a function of $\nu \in (0, \beta]$. Differentiating (3.8) with respect to ν, and using again (3.8) in the resulting equation to eliminate ν^2, it is found that

$$\frac{\partial r^2}{\partial \nu} = \frac{4\omega^2 \nu r^5}{\left[r^4 + 3(1-\omega)^2\right]\left[r^4 - (1-\omega)^2\right]} > 0. \tag{3.11}$$

This implies that max r or, in turn, $\max|z_1(\omega)| = \max|z_2(\omega)|$ is achieved for $\nu = \beta = \rho(T)$. So, putting β instead of ν in (3.8), that is, considering

$$\left[r^2 - (1-\omega)\right]^2\left[r^2 + (1-\omega)\right] - \omega^2 r^3 \beta^2 = 0, \tag{3.8$'$}$$

and repeating exactly the same argumentation as before, we end up with (3.10) again; the only difference now is that r refers to $\max|z_j(\omega, \beta)|$, $j = 1, 2$. Rewriting (3.8′) in the form

$$h(\omega) := r^3\beta^2 - \left(\frac{r^2 - (1-\omega)}{\omega}\right)^2\left[r^2 + (1-\omega)\right] = 0, \tag{3.12}$$

and using (3.10), it is readily obtained that

$$\begin{aligned} \operatorname{sign} h(0) &= \operatorname{sign}\Big(\lim_{\omega \to 0^+} h(\omega)\Big) = \operatorname{sign}(\beta^2 - 2) \\ \operatorname{sign} h(1) &= \operatorname{sign}\left(\beta^2 - \frac{1}{\sqrt{8}}\right). \end{aligned} \tag{3.13}$$

Since, on the other hand, it can be found from (3.10) that $\partial r/\partial\omega < 0$ and from (3.12), after a modest amount of algebra, that $\partial h/\partial\omega > 0$ and that $\partial\omega/\partial\beta < 0$, it is concluded that $r = \max|z_j(\omega, \beta)|^2$ (< 1), $j = 1, 2$, in (3.10) is minimized:

(1) for $0 < \beta \leqslant 1/\sqrt[4]{8}$ when $\omega = \omega_d = 1$ ($r = \beta^{2/3} < 1$),

(2) for $1/\sqrt[4]{8} < \beta < \sqrt{2}$ when $\omega = \omega_d$, the unique real root of (3.12) or of (3.8′) in (0, 1) [$r < 1$ is given in (3.10)], and

(3) for $\beta \geqslant \sqrt{2}$ when $\omega = \omega_d = 0$ (in which case $r = 1$).

It remains to be proven that if $\omega_d \in (\omega_c, 1)$ then $|z_3(\omega_c)| < |z_1(\omega_c)| = r^{1/2}$, where z_3 is the absolutely larger of the two real negative zeros z_3 and z_4 of

(2.1) or of (3.6). From (3.6) one obtains, for $\omega = \omega_d$, that

$$z_3 + z_4 = -\frac{\left(r^2 + 1 - \omega_d\right)^{1/2}}{r^{1/2}},$$
$$z_3 z_4 = \frac{1 - \omega_d}{r}. \tag{3.14}$$

Hence z_3 and z_4 are the roots of the quadratic

$$Z^2 + \frac{\left(r^2 + 1 - \omega_d\right)^{1/2}}{r^{1/2}} Z + \frac{1 - \omega_c}{r} = 0. \tag{3.15}$$

The modulus of the absolutely largest root of (3.15) is given by

$$| z_3 | = \frac{1}{2}\left\{\frac{\left(r^2 + 1 - \omega_d\right)^{1/2} + \left[r^2 - 3\left(1 - \omega_d\right)\right]^{1/2}}{r^{1/2}}\right\}, \tag{3.16}$$

where, obviously ,$r^2 \geqslant 3(1 - \omega_d)$, since z_3 and z_4 are real. A straightforward comparison shows that $r^{1/2} = | z_1 | > | z_3 |$ at $\omega = \omega_d$. Consequently $\omega_{\text{opt}} = \omega_d$, which concludes the proof in the present case $p = 4$ and therefore that of Theorem 1.

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