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On equivalence of three-parameter iterative methods for singular symmetric saddle-point problem

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Abstract
There have been a couple of papers for the solution of the nonsingular symmetric saddle-point problem using three-parameter iterative methods. In most of them, regions of convergence for the parameters are found, while in three of them, optimal parameters are determined, and in one of the latter, many more cases, than in all the others, are distinguished, analyzed, and studied. It turns out that two of the optimal parameters coincide making the optimal three-parameter methods be equivalent to the optimal two-parameter known ones. Our aim in this work is manifold: (i) to show that the iterative methods we present are equivalent, (ii) to slightly change some statements in one of the main papers, (iii) to complete the analysis in another one, (iv) to explain how the transition from any of the methods to the others is made, (v) to extend the iterative method to cover the singular symmetric case, and (vi) to present a number of numerical examples in support of our theory. It would be an omission not to mention that the main material which all researchers in the area have inspired from and used is based on the one of the most cited papers by Bai et al. (Numer. Math. 102:1–38, 2005).

Keywords Nonsingular/singular symmetric saddle-point problem · Three-parameter iterative solution methods · Optimal parameters · Optimal semi-convergence factor

Mathematics subject classification (2010) Primary 65F10. Secondary 65F08

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1 Introduction

Let the nonsingular symmetric saddle-point problem be defined by the linear system

$$\mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} A & B \\ -B^T & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ -q \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is symmetric positive definite, $B \in \mathbb{R}^{m\times n}$, $m > n$, is of full rank (rank$(B) = n$), $(\cdot)^T$ denotes transpose, and $x, p \in \mathbb{R}^m$, $y, q \in \mathbb{R}^n$.

Linear system (1.1) arises in various scientific and engineering applications as, e.g., in weighted least-squares problems, finite element discretization of the Navier-Stokes equation, constrained optimization, computer graphics, electronic networks, etc.; for an account of these applications as well as for related references, see, e.g., [32]. Solutions for special cases of the linear system (1.1) have been proposed by many researchers. We mention some of the main works based on extensions and generalizations of the classical iterative methods like SOR, SSOR, and MSOR (see, e.g, Varga [26] or Young [31]). The first work in the twenty-first century is the one by Golub et al. [10], where the SOR-like method was introduced. Here, we simply mention some of the main works in the area in the last eighteen years: Golub et al. [10], Bai et al. [2], Darvishi and Hessari [8], Bai and Wang [4], Wu et al. [27], Zheng et al. [32], Zhang and Wei [36], Zhang et al. [33], Zhang and Shen [35], Zhou and Zhang [38], Cao et al. [6], Louka and Missirlis [20], Njeru and Guo [25], Hadjidimos [14, 15], Huang and Wang [18], Feng et al. [9], Guo and Hadjidimos [11], etc. We mention that in the work by Golub et al. [10], an excellent account of the works prior to 2001 can be found and also an account of the works until 2009 can be found in Zheng et al. [32].

In Section 2, we present four methods. In Section 3, we show the equivalence of the four methods presented in the previous section by briefly exhibiting a one-to-one correspondence of the last three methods to the first one. In Section 4, we move on to the solution of the singular analogue to (1.1). In Section 5, a number of examples are presented in support of the theory developed. Finally, in Section 6, we make a number of concluding remarks.

2 Three-parameter iterative methods for the solution of (1.1)

In their monumental work [2], Bai, Parlett, and Wang introduced in Section 7, what they called *generalized inexact accelerated overrelaxation* (GIAOR) iterative method for the solution of the problem (1.1). This method contained two real matrix parameters $P \approx A$ and $Q \approx B^T A^{-1} B$ and three real parameters ω, τ, and γ. In the same Section 7 of [2], a "simplified" version of it for $P = A$ was considered, renamed later by Bai and Wang *accelerated parameterized inexact Uzawa* (APIU) iterative method. So, what we are to consider in the next subsections is the iterative solution of the nonsingular symmetric saddle-point problem by the APIU iterative method using three parameters, instead of the usual two, ω and τ, the main seed of which regarding their intervals of convergence can be found in the aforementioned Section 7 of [2].

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Following up the APIU iterative method, another three iterative methods are considered (maybe more have appeared in the literature). We make a number of comments on each of them, we point out what their strong points are and make modifications and improvements to the last two methods, respectively.

First, we present (i) the APIU method by Bai et al. [2]; (ii) each of the two methods by Louka and Missirlis [20] (see also [19]); (iii) the technique for APIU iterative method by Huang and Wang [18], where many extensions in various directions are distinguished, analyzed, and studied by the authors (see next paragraph and Remarks in the beginning of Section 2.3), and some statements in it are slightly modified; and (iv) finally, the iterative method by Feng et al. [9] which will be completed. Secondly, it is indirectly shown that all the four methods are equivalent and so the parameters of each one of the last four can be expressed in terms of those in [2]. Note that optimal parameters have been determined only in the works by Louka and Missirlis [20] (see also [19]) in the classical case of [2], while by Huang and Wang [18] optimal values were obtained in two distinct cases. Hence, these optimal values can be carried over to the other two works. We point out that the main characteristic of the optimal parameters is that two of them coincide and so the optimal three-parameter iterative methods for the solution of problem (1.1) make these optimal problems be identical with all the equivalent optimal two-parameter ones that were analyzed and studied in Hadjidimos [15].

It would be an omission on our part if we did not explicitly mention what the work by Huang and Wang [18] contributed to the APIU iterative method: (i) The "optimal parameters" were determined using pure analysis. (ii) The authors determined "optimal parameters" also for $m = n$ a case "overlooked" by previous researchers. (iii) They determined "regions of convergence" and "optimal parameters" for $m \geq n$ not only when the iteration matrix involved has a positive spectrum but also when the corresponding spectrum is negative. (iv) Finally, they presented in Table 1 the "Possible optimum point(s) for Uzawa-like methods discussed in (their) Theorem 6.1" which gives the idea of equivalence of relevant methods.

2.1 Bai-Parlett-Wang's three-parameter iterative method [2]

The *accelerated parameterized inexact Uzawa* (APIU) iterative method can be constructed as follows. First, the splitting

$$\mathcal{A} := \begin{bmatrix} A & B \\ -B^T & 0 \end{bmatrix} = \mathcal{D} - \mathcal{L} - \mathcal{U} \tag{2.1}$$

is considered, where

$$\mathcal{D} = \begin{bmatrix} A & 0 \\ 0 & Q \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} 0 & 0 \\ B^T & 0 \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} 0 & -B \\ 0 & Q \end{bmatrix} \tag{2.2}$$

and $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and an approximation to the Schur complement $B^T A^{-1} B$ of the matrix $\mathcal{A}$. Next, two diagonal matrices are considered containing three nonzero real parameters

$$\Omega = \begin{bmatrix} \omega I_m & 0 \\ 0 & \tau I_n \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & \gamma I_n \end{bmatrix} \tag{2.3}$$

and the block AOR-type iterative method (see [13]) given below is proposed

$$\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = (\mathcal{D} - R\mathcal{L})^{-1}[(I_{m+n} - \Omega)\mathcal{D} + (\Omega - R)\mathcal{L} + \Omega\mathcal{U}] \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} + (\mathcal{D} - R\mathcal{L})^{-1}\Omega \begin{bmatrix} p \\ -q \end{bmatrix} \tag{2.4}$$

or, equivalently,

$$\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \mathcal{T}(\omega, \tau, \gamma) \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} + \mathcal{F}^{-1}(\omega, \tau, \gamma) \begin{bmatrix} p \\ -q \end{bmatrix}, \tag{2.5}$$

where

$$\begin{aligned} \mathcal{T}(\omega, \tau, \gamma) &:= (\mathcal{D} - R\mathcal{L})^{-1}[(I_{m+n} - \Omega)\mathcal{D} + (\Omega - R)\mathcal{L} + \Omega\mathcal{U}] \\ &= \begin{bmatrix} (1-\omega)I_m & -\omega A^{-1}B \\ (\tau - \omega\gamma)Q^{-1}B^T & I_n - \omega\gamma Q^{-1}B^T A^{-1}B \end{bmatrix} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \mathcal{F}(\omega, \tau, \gamma) &:= \Omega^{-1}(\mathcal{D} - R\mathcal{L}) = \begin{bmatrix} \frac{1}{\omega}A & 0 \\ -\frac{\gamma}{\tau}B^T & \frac{1}{\tau}Q \end{bmatrix}, \\ \mathcal{F}^{-1}(\omega, \tau, \gamma) &= \begin{bmatrix} \omega A^{-1} & 0 \\ \omega\gamma Q^{-1}B^T A^{-1} & \tau Q^{-1} \end{bmatrix}. \end{aligned} \tag{2.7}$$

Hence, the APIU method (2.4), using (2.5)–(2.7), can be written as

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega A^{-1}(p - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \tau Q^{-1}(B^T x^{(k)} - q) + \gamma Q^{-1}B^T(x^{(k+1)} - x^{(k)}), \end{cases} \tag{2.8}$$

with any $\left[x^{(0)T} \; y^{(0)T}\right]^T \in \mathbb{R}^{m+n}$ and $k = 0, 1, 2, \cdots$.

The remark below makes clear a crucial point of the main statement that follows it.

Remark 2.1 Let λ be an eigenvalue of the iteration matrix $\mathcal{T}(\omega, \tau, \gamma)$ in (2.6) and $z := \left[x'^T \; y'^T\right]^T \in \mathbb{R}^{m+n}$ be its associated eigenvector. From $\mathcal{T}(\omega, \tau, \gamma)z = \lambda z$, we obtain

$$\begin{aligned} &(\lambda + \omega - 1)x' + \omega A^{-1}By' = 0, \\ &(\omega\gamma - \tau)Q^{-1}B^T x' + \left((\lambda - 1)I_n + \omega\gamma Q^{-1}B^T A^{-1}B\right) y' = 0. \end{aligned} \tag{2.9}$$

Then, if $\lambda = 1 - \omega$ from the first relation in (2.9), in view of B being of full rank, implies that $y' = 0$ and from the second relation we will have, for $\omega\gamma \neq \tau$, $x' \in \mathcal{N}(B^T)$, where $\mathcal{N}(\cdot)$ denotes *nullspace*. Hence, the eigenvalue $\lambda = 1 - \omega$ will have as associated eigenvector $z = [x'^T, \, 0_{m-n}^T]^T$, with $x' \in \mathcal{N}(B^T)$. If $\tau = \omega\gamma$, then x' may be any vector in $\mathbb{R}^m \backslash \{0\}$ and the three-parameter method becomes a two-parameter one.

Below we present in a very condensed form a statement that contains the results of Theorems 7.1 and 7.2(i) of [2].

Theorem 2.1 *(Theorems 7.1 and 7.2(i) of* [2]*) Under the notations and assumptions so far, let* μ *be an eigenvalue of* $\mathcal{J} = Q^{-1}B^T A^{-1} B$ $(\mu \in \sigma(\mathcal{J}))$. *Then,* $\lambda \in \sigma(\mathcal{T}(\omega, \tau, \gamma))$ *implies that either* $\lambda = 1 - \omega$ *or* λ *is a root of the quadratic equation*

$$\lambda^2 - (2 - \omega - \omega\gamma\mu)\lambda + (1 - \omega) + \omega(\tau - \gamma)\mu = 0 \tag{2.10}$$

and the APIU iterative method (2.8) *converges if and only if*

$$\omega \in (0,\ 2)\,, \quad \tau \in \left(0,\ \frac{4}{\omega\mu_{\max}}\right), \quad \gamma \in \left(\tau - \frac{1}{\mu_{\max}},\ \frac{\tau}{2} + \frac{2-\omega}{\omega\mu_{\max}}\right), \tag{2.11}$$

where $\mu_{\min}$ *and* $\mu_{\max}$ *are the smallest and the largest eigenvalues of the matrix* $\mathcal{J} = Q^{-1}B^T A^{-1} B$.

It should be noted that the conditions under which all the zeros of a complex polynomial are within the unit circle can be determined by the Schur-Cohn algorithm (see, e.g., Vol. 1, p. 493 of Henrici [16]). This has been used by many authors before, e.g., by J.H.H. Miller [24] for the location of the zeros of certain classes of polynomials, etc. We would also like to note that the roots of complex quadratic polynomials were described in the proof of Theorem 4.3 in Bai et al. [2] and the roots of complex cubic polynomial equation were described in a recent work by Z.-Z. Bai and M. Tao (see Lemma 3.2 in [3]).

However, in order to find the conditions (2.11) under which the roots $\lambda \in \sigma(\mathcal{J})$ of the monic real quadratic (2.10) are strictly less than 1 in modulus, one may use Lemma 2.1, pp. 171–172 of Young [31] presented in the sequel.

Lemma 2.1 *If* b *and* c *are real, then both roots of the quadratic equation*

$$x^2 - bx + c = 0 \tag{2.12}$$

are less than one in modulus if and only if

$$|c| < 1, \quad |b| < 1 + c. \tag{2.13}$$

2.2 Louka-Missirlis's three-parameter iterative methods [20]

Louka and Missirlis [20] (see also [19]) were the first researchers who determined the optimal parameters of the three-parameter optimal APIU iterative method introduced by Bai et al. in [2] and presented in Section 2.1 before. In fact, they proposed two iterative methods called *generalized modified extrapolated SOR* (GMESOR) and *generalized modified preconditioned simultaneous displacement* (GMPSD) which are presented in the following two subsections. It is interesting to note that the optimal solution was found by an ingenious combination of an algebraic and a geometric method the latter of which was inspired by that given on pp. 123–125 of Varga's book [26] for the determination of the optimal overrelaxation parameter of a two-cyclic SOR method when the associated Jacobi iteration matrix is weakly-cyclic of index 2 and the squares of its eigenvalues are nonnegative and strictly less than one. In the sequel, the two methods are presented in "simplified" versions and the various entities, except the parameters involved, are denoted by the same symbols as those of Section 2.1.

2.2.1 The generalized modified extrapolated SOR (GMESOR) method

First, the splitting (2.1) is considered but the components of $\mathcal{A}$ are a little different from the previous ones since one nonzero real parameter a is introduced as is shown below

$$\mathcal{D} = \begin{bmatrix} A & 0 \\ 0 & Q \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} 0 & 0 \\ B^T & aQ \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} 0 & -B \\ 0 & (1-a)Q \end{bmatrix}, \tag{2.14}$$

where Q is the matrix considered in (2.2). To be consistent with the notation in Section 2.1, we introduce nonzero diagonal matrices

$$\Omega = \begin{bmatrix} \tau_1 I_m & 0 \\ 0 & \tau_2 I_n \end{bmatrix}, \quad R = \begin{bmatrix} \omega_1 I_m & 0 \\ 0 & \omega_2 I_n \end{bmatrix}. \tag{2.15}$$

It seems that from this point onwards the authors follow exactly the same process as in Section 2.1 except that two of the three parts $\mathcal{D}, \mathcal{L}, \mathcal{U}$ of $\mathcal{A}$ are different. Hence, they end up with the algorithm for their GMESOR method given in an analogous way to that in (2.8). Specifically,

$$\begin{cases} x^{(k+1)} = (1-\tau_1)x^{(k)} + \tau_1 A^{-1}(p - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \frac{\tau_2}{1-a\omega_2} Q^{-1}(B^T x^{(k)} - q) + \frac{\omega_2}{1-a\omega_2} Q^{-1} B^T (x^{(k+1)} - x^{(k)}). \end{cases} \tag{2.16}$$

From (2.16), it is observed that the parameter ω_1 of the matrix R in (2.15) is not needed. Also, despite the presence of four parameters in (2.16), only three are practically used since the fractions $\frac{\tau_2}{1-a\omega_2}$ and $\frac{\omega_2}{1-a\omega_2}$ play the roles of τ and γ, respectively. So the parameter a becomes a "free" parameter. Evidently, the GMESOR method is identical with the APIU method of Bai et al. [2] with coincidence of their relevant parameters as follows:

$$\tau_1 = \omega, \quad \tau_2 = \tau(1-a\omega_2), \quad \omega_2 = \gamma(1-a\omega_2) \;\Leftrightarrow\; \omega_2 = \frac{\gamma}{1+a\gamma}. \tag{2.17}$$

Remark 2.2 If one wishes to use the nonzero parameter a as the authors of [20] did, one may use all the coordinate pairs (ω_2, a) of the (ω_2, a)-plane except those lying on the axes and on the hyperbola $a\omega_2 = 1$ and those that do not guarantee convergence of the GMESOR iterative method (see (2.17) and (2.11)). It should be pointed out that some of these observations were also made by the authors of [20]. In addition, it is noted that ω, τ, and γ are found in terms of $\tau_1, \tau_2, \omega_2, a$; however, when $a = 0$, the parameters of Bai et al. [2] coincide with those of Louka and Missirlis's [20].

A statement that gives the analogous to (2.10) functional equation for the eigenvalues of the iteration matrix is presented below.

Theorem 2.2 *(Theorem 2.1 of* [20]*) Under the notation, assumptions, and restrictions so far, let* μ *be an eigenvalue of* $\mathcal{J} = Q^{-1}B^TA^{-1}B$ $(\mu \in \sigma(\mathcal{J}))$. *Then,* $\lambda \in \sigma(\mathcal{T}(\tau_1, \tau_2, \omega_2, a))$ *implies that either* $\lambda = 1 - \tau_1$ *or* λ *is a root of the quadratic equation*

$$\lambda^2 - \left(2 - \tau_1 - \frac{\tau_1\omega_2}{1 - a\omega_2}\mu\right)\lambda + (1 - \tau_1) + \frac{\tau_1(\tau_2 - \omega_2)}{1 - a\omega_2}\mu = 0. \qquad (2.18)$$

Below we present Theorem 2.3 of [20] where the optimal parameters of the GMESOR iterative method are determined by a combination of an algebraic and geometric analysis as has already been mentioned. The end result is that the optimal parameters $\tau_{1_{opt}} = \omega_{2_{opt}}$ and the optimal three-parameter iterative method GMESOR or, equivalently, the APIU method reduces to the optimal two-parameter APIU method, namely the optimal "generalized successive overrelaxation (GSOR)" method of Bai et al.'s [2]. More specifically:

Theorem 2.3 *(Theorem 2.3 of* [20]*) Under the assumptions so far and the main assumptions of Theorem 2.1, the optimal three-parameter method GMESOR has* $\tau_{2_{opt}} = \omega_{2_{opt}}$ *and so it coincides with the optimal two-parameter APIU method of Bai et al.* [2]. *Hence, the optimal parameters of GMESOR are*

$$\begin{aligned} \omega_{2_{opt}} = \tau_{2_{opt}} = \frac{1}{a+\sqrt{\mu_{\max}\mu_{\min}}}, \quad \tau_{1_{opt}} = \frac{4\sqrt{\mu_{\max}\mu_{\min}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2}, \\ \rho\left(\mathcal{T}(\tau_{1_{opt}}, \tau_{2_{opt}}, \omega_{2_{opt}}, a)\right) = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}. \end{aligned} \qquad (2.19)$$

Furthermore, if one sets the "free" parameter $a = 0$, *then the GMESOR method reduces to the APIU method whose optimal parameters are*

$$\begin{aligned} \tau_{opt} = \gamma_{opt} = \frac{1}{\sqrt{\mu_{\max}\mu_{\min}}}, \quad \omega_{opt} = \frac{4\sqrt{\mu_{\max}\mu_{\min}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2}, \\ \rho\left(\mathcal{T}(\omega_{opt}, \tau_{opt}, \gamma_{opt})\right) = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}. \end{aligned} \qquad (2.20)$$

Remark 2.3 If $\mu_{\min}\mu_{\max}$ is sufficiently small, the parameter a can be used as a regularization parameter to make the computation of the optimal parameters $\omega_{2_{opt}}$ and $\tau_{2_{opt}}$ be more stable if, of course, they are computable. Although the optimal spectral radius of the GMESOR method is identical with that of the optimal APIU method in the former method the "free" parameter a was introduced in the hope to accelerate the Krylov subspace methods. For example, as a preconditioning matrix of $\mathcal{A}$ in (1.1) for the GMRES method, one may take $(\mathcal{D} - R\mathcal{L})^{-1}\,\Omega$, where $\mathcal{D}$, R, $\mathcal{L}$, and Ω are given in (2.14)–(2.15) with $\omega_1, \omega_2 = \tau_2$ their optimal values from (2.19) for various $a \in (0, 1)$ close to 0.

2.2.2 The generalized modified simultaneous displacement (GMPSD) method

The method in the title, which the authors of [20] (see also [19]) considered and studied in detail, is much more complicated and much lengthier than their GMESOR method. In the present authors' opinion, the GMPSD method was studied and analyzed in the hope that a better optimal method than the previous one would be obtained. So in what follows, we are to give only some of the main parts and results of it and for the rest the reader is referred to [20]. As in the two previous cases, we give the main splitting of $\mathcal{A}$ into its three parts (diagonal, strictly lower triangular, strictly upper triangular) and the diagonal matrices Ω and R are the same as before and are given in (2.14) and (2.15), respectively. The main difference is that the preconditioning matrix is now $T^{-1}(\mathcal{D}-\Omega\mathcal{L})\mathcal{D}^{-1}(\mathcal{D}-\Omega\mathcal{U})$ instead of $T^{-1}(\mathcal{D}-\Omega\mathcal{L})$ used before.

So, after the construction of the iterative method one ends up with the GMPSD algorithm which is as follows:

$$\left\{\begin{array}{rl} y^{(k+1)} &= y^{(k)} + \frac{1}{(1-a\omega_2)[1-(1-a)\omega_2]} Q^{-1}\left\{B^T\left[(\tau_2-\tau_1\omega_2)x^{(k)}\right.\right. \\ &+ \left.\left.\tau_1\omega_2 A^{-1}(p-By^{(k)})\right]-\tau_2 q\right\}, \\ x^{(k+1)} &= (1-\tau_1)x^{(k)} + A^{-1}\{B[(\omega_1-\tau_1)y^{(k)}-\omega_1 y^{(k+1)}]+\tau_1 p\}. \end{array}\right. \tag{2.21}$$

The eigenvalues of the iteration matrix $\mathcal{T}(\omega_1,\omega_2,\tau_1,\tau_2,a)$ are $\lambda = 1-\tau_1$ and the rest of them are given by the roots of the functional equation

$$\lambda^2 - \left(2-\tau_1-\frac{\tau_1\omega_2+\tau_2\omega_1-\tau_1\omega_1\omega_2}{(1-a\omega_2)[1-(1-a)\omega_2]}\mu\right)\lambda + (1-\tau_1) + \frac{\tau_1\tau_2+\tau_2\omega_1-\tau_1\omega_1\omega_2-\tau_1\omega_2}{(1-a\omega_2)[1-(1-a)\omega_2]}\mu = 0. \tag{2.22}$$

Finally, the optimal parameters found in [20] are given by the expressions below.

Theorem 2.4 *(Theorem 3.3 of* [20]*) Under the notation, assumptions, and restrictions so far and the additional restriction* $\omega_2 \neq \frac{\tau_{2_{opt}}}{\tau_{1_{opt}}}$*, the optimal parameters of the GMPSD method are as follows*

$$\begin{array}{c} \tau_{1_{opt}} = \frac{4\sqrt{\mu_{\max}\mu_{\min}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2}, \quad \tau_{2_{opt}} = \frac{(1-a\omega_2)[1-(1-a)\omega_2]}{\sqrt{\mu_{\max}\mu_{\min}}}, \\ \omega_{1_{opt}} = \frac{\tau_{1_{opt}}(\tau_{2_{opt}}-\omega_2)}{\tau_{2_{opt}}-\tau_{1_{opt}}\omega_2}, \\ \rho\left(\mathcal{T}(\tau_{1_{opt}},\tau_{2_{opt}},\omega_{1_{opt}},a,\omega_2)\right) = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}. \end{array} \tag{2.23}$$

Remark 2.4 The optimal spectral radius of the GMPSD method is identical with that of the optimal APIU method. However, in the former, there are two "free" parameters a and ω_2 which may be useful in accelerating the Krylov subspace methods. Observe that for $\omega_2 = 0$ and any a, the optimal GMPSD method becomes the optimal APIU method.

2.3 Huang-Wang's APIU iterative method [18]

To the best of our knowledge, the only other researchers who have determined not only the regions of convergence of the three parameters involved in the class of methods we are studying but also their optimal parameters by purely analytical methods are Huang and Wang [18]. Their convergence regions and the optimal parameters are the same as those of the GMESOR method with $a = 0$. The authors followed and extended the analysis of Theorems 7.1 and 7.2(i) of Bai et al. [2] by keeping the same notation, succeeded in extending it in various directions, and solved completely the problem of the determination of the optimal parameters.

For the solution of the original problem in (1.1), it is natural to know something about the entities involved in it and especially the matrix A. All other authors who we are talking about in this paper considered A to be real symmetric positive definite and so is the parameter matrix Q implying that the matrix $\mathcal{J} := Q^{-1}B^TA^{-1}B$ has real positive eigenvalues. (Note that at the same time for A and Q real symmetric negative definite the corresponding (1.1) problem has precisely the same intervals of convergence and the same optimal parameters.) For A real symmetric negative definite and Q real symmetric positive definite (or the other way around) leading to the corresponding matrix $\mathcal{J}$ having real negative eigenvalues nobody had dealt with so far.

It is worth pointing out that Huang and Wang [18], besides the issue described in the previous paragraph, dealt with one more which led them to distinguish, analyze, and study many more cases, which are presented very briefly in the following two remarks:

Remark 2.5 They considered separately the case of positive eigenvalues for $\mathcal{J} = Q^{-1}B^TA^{-1}B$ ($\sigma(\mathcal{J}) \subset [\mu_{\min}, \mu_{\max}] \subset (0, +\infty)$) and that of negative eigenvalues for $\mathcal{J}$ ($\sigma(\mathcal{J}) \subset [-\mu_{\max}, -\mu_{\min}] \subset (-\infty, 0)$) and ended up with two sets of convergence regions for the three parameters as well as for the optimal ones (see relations (4) and Sections 5 and 6 of [18] to which the interested reader is strongly recommended).

Remark 2.6 To the best of the present authors' knowledge, all the researchers had treated the case $m \geq n$ (or have ignored the equality sign), with B of full rank, as being one case and ended up with a set of regions of convergence. However, when the two cases $m > n$ and $m = n$ are studied separately, as Huang and Wang did in [18] (see relations (5), (6), etc.), they ended up with two sets of different results. It seems that it was the first time where such a distinction had been made. It should also be pointed out that although the optimal parameters for $m > n$ and $m = n$ were found to be the same, the regions of convergence were not only much wider in the latter case but also there were two different sets of them. This is because, as Huang and Wang shown, for $m > n$, the restriction $|\omega - 1| < 1$ constitutes among others one of the necessary conditions for convergence, while for $m = n$ such a restriction does not exist!

In the sequel, we present a statement that finds the exact regions of convergence of Theorem 1 of Huang and Wang [18], where some slight modifications have been made. Below and without loss of generality, we take the case A real symmetric positive definite and Q of the same or of the opposite definiteness.

Theorem 2.5 *(Slightly modified version of Huang and Wang's Theorem 1* [18]*) Let A, Q, $B^TA^{-1}B$ be nonsingular and real symmetric and the eigenvalues $\mu \in \sigma(\mathcal{J}) = \sigma(Q^{-1}B^TA^{-1}B)$ be all of the same sign, with $\mu_{\max}$ and $\mu_{\min}$ denoting the largest and the smallest eigenvalues in modulus. Then, the APIU iterative method coincides with* (2.16) *for $a = 0$, where the parameters involved ω, τ, γ are those in* (2.17)*, and*

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converges to the unique solution of (1.1) *for any initial choice of* $\left[x^{(0)T}\ y^{(0)T}\right]^T \in \mathbb{R}^{m+n}$ *if and only if its parameters satisfy:*

For $m > n$

$$\begin{cases} \omega \in (0,2),\ \tau > 0,\ \gamma \in \left(\tau - \frac{1}{\mu_{\max}},\ \frac{\tau}{2} + \frac{2-\omega}{\omega\mu_{\max}}\right), & [\mu_{\min}, \mu_{\max}] \subset (0, +\infty), \\ \omega \in (0,\ 2)\,,\ \tau < 0,\ \gamma \in \left(\frac{\tau}{2} - \frac{2-\omega}{\omega\mu_{\max}},\ \tau + \frac{1}{\mu_{\max}}\right), & [-\mu_{\max}, -\mu_{\min}] \subset (-\infty, 0). \end{cases} \tag{2.24}$$

For $m = n$

$$\begin{cases} [\mu_{\min}, \mu_{\max}] \subset (0, +\infty), \\ \omega > 0, \quad \tau > 0, \quad \gamma \in \left(\tau - \frac{1}{\mu_{\max}},\ \begin{Bmatrix} \frac{\tau}{2} + \frac{2-\omega}{\omega\mu_{\max}} & \textit{for } \omega \in (0,2) \\ \frac{\tau}{2} & \textit{for} \quad \omega = 2 \\ \frac{\tau}{2} + \frac{2-\omega}{\omega\mu_{\min}} & \textit{for} \quad \omega > 2 \end{Bmatrix}\right), \\ \omega < 0, \quad \tau < 0, \quad \gamma \in \left(\frac{\tau}{2} + \frac{2-\omega}{\omega\mu_{\max}},\ \tau - \frac{1}{\mu_{\min}}\right), \end{cases} \tag{2.25}$$

$$\begin{cases} [-\mu_{\max}, -\mu_{\min}] \subset (-\infty, 0), \\ \omega > 0, \quad \tau < 0, \quad \gamma \in \left(\begin{Bmatrix} \frac{\tau}{2} - \frac{2-\omega}{\omega\mu_{\max}} & \textit{for } \omega \in (0,2) \\ \frac{\tau}{2} & \textit{for} \quad \omega = 2 \\ \frac{\tau}{2} - \frac{2-\omega}{\omega\mu_{\min}} & \textit{for} \quad \omega > 2 \end{Bmatrix},\ \tau + \frac{1}{\mu_{\max}}\right), \\ \omega < 0, \quad \tau > 0, \quad \gamma \in \left(\tau + \frac{1}{\mu_{\min}},\ \frac{\tau}{2} - \frac{2-\omega}{\omega\mu_{\max}}\right). \end{cases} \tag{2.26}$$

Proof For $m > n$

The first case in (2.24) is nothing but the one where the intervals for the three parameters are given in (2.11). Now, to see how easy it is to find the regions of convergence for Q negative definite from the ones for Q positive definite consider the iteration matrix $\mathcal{T}(\omega, \tau, \gamma)$ in (2.6) and write it as

$$\begin{aligned} \mathcal{T}(\omega, \tau, \gamma) &:= (\mathcal{D} - R\mathcal{L})^{-1}[(I_{m+n} - \Omega)\mathcal{D} + (\Omega - R)\mathcal{L} + \Omega\mathcal{U}] \\ &= \begin{bmatrix} (1-\omega)I_m & -\omega A^{-1}B \\ ((-\tau) - \omega(-\gamma))(-Q)^{-1}B^T & I_n - \omega(-\gamma)(-Q)^{-1}B^TA^{-1}B \end{bmatrix}. \end{aligned} \tag{2.27}$$

Observe now that changing the signs of the parameters τ and γ and the matrix Q in the last row of the matrix in (2.6) produces the identical matrix $\mathcal{T}(\omega, \tau, \gamma)$ in (2.27). But since $-Q$ is positive definite, and $\mathcal{J} := Q^{-1}B^TA^{-1}B$, the spectrum of $-\mathcal{J}$ is also positive definite and satisfies $\sigma(-\mathcal{J}) \subset [\mu_{\min}, \mu_{\max}] \subset (0, +\infty)$ implying $\sigma(\mathcal{J}) \subset [-\mu_{\max}, -\mu_{\min}] \subset (-\infty, 0)$. Note also that the aforementioned change of signs does not change the algorithm (2.5).

Therefore, the intervals of convergence of the parameters involved in (2.11) are as follows:

$$\omega \in (0,\ 2)\,, \quad -\tau \in \left(0,\ \frac{4}{\omega\mu_{\max}}\right), \quad -\gamma \in \left(-\tau - \frac{1}{\mu_{\max}},\ \frac{-\tau}{2} + \frac{2-\omega}{\omega\mu_{\max}}\right). \tag{2.28}$$

Changing the signs in the last two inclusions, we end up with the second relations of (2.24).

For $m = n$

In this case, the eigenvalues of the iteration matrix $\mathcal{T}(\omega, \tau, \gamma)$ come entirely from the functional (2.10) employing Lemma 2.1; the restriction $\omega \in (0, 2)$ does not apply any more because there are no eigenvalues equal to $1 - \omega$ unless $\tau = \omega\gamma$ in which case the three-parameter iterative method becomes a two-parameter one. However, a more detailed analysis is needed regarding relations (5) and (6) of [18] because ω and τ cannot change independently of each other in their intervals of convergence as is shown below.

Take for example the first relation in (5) of [18]. This comes from the relations of Lemma 2.1 considering that $\mu \in \sigma(\mathcal{J}) \subset [\mu_{\min}, \mu_{\max}] \subset (0, +\infty)$. Using (2.10) and Lemma 2.1 and getting rid of the absolute values of relations (2.13), we end up with the following four new relations:

$$\begin{gathered} -1 < (1-\omega) + \omega(\tau - \gamma)\mu < 1, \\ -1 - (1-\omega) - \omega(\tau-\gamma)\mu < 2 - \omega - \omega\gamma\mu < 1 + (1-\omega) + \omega(\tau-\gamma)\mu. \end{gathered} \tag{2.29}$$

The very last relation gives $\omega\tau > 0$ and so we distinguish two cases. Hence, for $\omega > 0$ and $\tau > 0$, the rightmost inequality of the first two in (2.29) and the leftmost inequality of the last two lead to

$$\tau - \frac{1}{\mu} < \gamma < \frac{\tau}{2} + \frac{2-\omega}{\omega\mu}. \tag{2.30}$$

However, for the leftmost expression to be strictly less than the rightmost one, there must hold

$$\tau - \frac{1}{\mu} < \frac{\tau}{2} + \frac{2-\omega}{\omega\mu} \iff \tau < \frac{4}{\omega\mu} \implies \tau \in \left(0, \frac{4}{\omega\mu_{\max}}\right). \tag{2.31}$$

The results in (2.30)–(2.31) and the fact that the parameter ω can be $\omega \in (0, 2),\ \omega = 2,\ \omega \in (2, +\infty)$, lead one to obtain the first set of intervals of convergence for the triad (ω, τ, γ) in (2.25).

Similarly, if $\omega < 0$ and $\tau < 0$, we can find the second set of the intervals for the same parameters in (2.25).

Now, consider the case where $\mu \in \sigma(\mathcal{J}) \subset [-\mu_{\max}, -\mu_{\min}] \subset (-\infty, 0)$, with $\mu_{\max} > 0$, and employ again Lemma 2.1. From the same relation, i.e., the very last inequality of (2.29), it turns out that the two parameters ω and τ satisfy $\omega\tau < 0$. Hence, we distinguish again the two cases $\omega > 0$, $\tau < 0$ and $\omega < 0$, $\tau > 0$. The aforementioned two cases are examined separately and, eventually, end up with the intervals for the triads of the parameters involved presented in (2.26).

The intervals in (2.25)–(2.26) give the complete list of relations of the four cases presented in (5)–(6) of [18] for $m = n$. □

Remark 2.7 (i) All the optimal results found in [18] are the same for $m > n$ and $m = n$.

(ii) When the eigenvalues μ of $\mathcal{J}$ are in $[\mu_{\min}, \mu_{\max}] \subset (0, +\infty)$, the optimal results of [18] are identical to those that had been obtained before by Louka and Missirlis [20] (see also [19]) for the GMESOR iterative method for $a = 0$. Huang and Wang [18] found by purely analytical methods the optimal results for both cases $\mu \in [\mu_{\min}, \mu_{\max}] \subset (0, +\infty)$ and $\mu \in [-\mu_{\max}, -\mu_{\min}] \subset (-\infty, 0)$ and presented them in their Lemma 11 and Theorem 2 and in a unified form in their Theorem 3.

(iii) When the eigenvalues μ of $\mathcal{J}$ are in $[-\mu_{\max}, -\mu_{\min}] \subset (-\infty, 0)$, the optimal results can be found directly by the technique used to go from (2.11) to (2.28) and then to (2.24).

Referring to items (ii) and (iii) of Remark 2.7, we may point out that it is not necessary to go through two different but similar analyses to find separately the optimal values for Q real symmetric positive definite and for Q real symmetric negative definite provided we use the technique mentioned before. Specifically, we may give the following statement.

Theorem 2.6 *Having found the optimal results for Q real symmetric positive definite in* [20] *presented in* (2.20)*, we may find directly the optimal results for Q real symmetric negative definite.*

Proof For $\mu \in [\mu_{\min}, \mu_{\max}] \subset (0, +\infty)$, then from (2.20), we have that

$$\omega_{opt} = \frac{4\sqrt{\mu_{\max}\mu_{\min}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2}, \quad \tau_{opt} = \gamma_{opt} = \frac{1}{\sqrt{\mu_{\max}\mu_{\min}}},$$
$$\rho\left(\mathcal{T}(\omega_{opt}, \tau_{opt}, \gamma_{opt})\right) = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}. \tag{2.32}$$

Using now the technique mentioned before, we have for the optimal results in the negative case the following. For $\mu \in [-\mu_{\max}, -\mu_{\min}] \subset (-\infty, 0)$, then

$$\omega_{opt} = \frac{4\sqrt{\mu_{\max}\mu_{\min}}}{\left(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}\right)^2}, \quad \tau_{opt} = \gamma_{opt} = -\frac{1}{\sqrt{\mu_{\max}\mu_{\min}}},$$
$$\rho\left(\mathcal{T}(\omega_{opt}, \tau_{opt}, \gamma_{opt})\right) = \left(1-\omega_{opt}\right)^{\frac{1}{2}} = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}. \tag{2.33}$$

Combining the optimal results in (2.32) and (2.33), we can give in both cases, as Huang and Wang [18] did in their Theorem 3, a unique form for the corresponding optimal values as this is repeated below:

$$\omega_{opt} = \frac{4\sqrt{\mu_{\max}\mu_{\min}}}{\left(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}\right)^2}, \quad \tau_{opt} = \gamma_{opt} = \frac{sgn(\mu)}{\sqrt{\mu_{\max}\mu_{\min}}},$$
$$\rho\left(\mathcal{T}(\omega_{opt}, \tau_{opt}, \gamma_{opt})\right) = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}, \tag{2.34}$$

where $\mu \in \sigma(\mathcal{J})$. □

2.4 Feng-Guo-Chen's three-parameter iterative method [9]

The last method we have known in the same area has been proposed by Feng, Guo, and Chen [9] very recently. It is called *modified accelerated successive overrelaxation* (MASOR) iterative method and constitutes an extension of the ASOR method introduced by Njeru and Guo [25]. The splitting of the matrix $\mathcal{A}$ into the three components $\mathcal{D}$, $\mathcal{L}$, $\mathcal{U}$ is based on

$$\mathcal{D}=\begin{bmatrix}\alpha A & 0\\ 0 & Q\end{bmatrix},\quad \mathcal{L}=\begin{bmatrix}-A & 0\\ B^T & \frac{1}{2}Q\end{bmatrix},\quad \mathcal{U}=\begin{bmatrix}\alpha A & -B\\ 0 & \frac{1}{2}Q\end{bmatrix},\tag{2.35}$$

with $Q\in\mathbb{R}^{n\times n}$ symmetric positive definite, and $\alpha>0$, $\omega\neq 0$ real parameters. (Note that in the present work we do not stick to the restriction for α but we let $\alpha\in\mathbb{R}\backslash\{0\}$.) The preconditioning matrix to be used is then $\frac{1}{\omega}(\mathcal{D}-\gamma\mathcal{L})=\begin{bmatrix}\frac{\alpha+\gamma}{\omega}A & 0\\ -\frac{\gamma}{\omega}B^T & \frac{2-\gamma}{2\omega}Q\end{bmatrix}$, with γ as a real parameter such that $(\alpha+\gamma)(2-\gamma)\neq 0$.

Using the above preconditioner, we can find

$$\begin{bmatrix}x^{(k+1)}\\ y^{(k+1)}\end{bmatrix}=(\mathcal{D}-\gamma\mathcal{L})^{-1}[(1-\omega)\mathcal{D}+(\omega-\gamma)\mathcal{L}+\omega\mathcal{U}]\begin{bmatrix}x^{(k)}\\ y^{(k)}\end{bmatrix}+\omega(\mathcal{D}-\gamma\mathcal{L})^{-1}\begin{bmatrix}p\\ -q\end{bmatrix},\tag{2.36}$$

which is nothing but a classical AOR-type method [12]. From (2.36), the iteration matrix of the MASOR iterative method, given in relations (3.1) of [9], is written as follows:

$$\mathcal{T}(\alpha,\omega,\gamma)=\begin{bmatrix}(1-\frac{\omega}{\alpha+\gamma})I_m & -\frac{\omega}{\alpha+\gamma}A^{-1}B\\ \frac{2\alpha\omega}{(\alpha+\gamma)(2-\gamma)}Q^{-1}B^T & I_n-\frac{2\omega\gamma}{(\alpha+\gamma)(2-\gamma)}Q^{-1}B^TA^{-1}B\end{bmatrix}.\tag{2.37}$$

Also, from (2.36), we can very easily construct the relevant algorithm of the MASOR iterative method which is

$$\begin{cases}x^{(k+1)}=(1-\frac{\omega}{\alpha+\gamma})x^{(k)}+\frac{\omega}{\alpha+\gamma}A^{-1}(p-By^{(k)}).\\ y^{(k+1)}=y^{(k)}+\frac{2\omega}{2-\gamma}Q^{-1}(B^Tx^{(k)}-q)+\frac{2\gamma}{2-\gamma}Q^{-1}B^T(x^{(k+1)}-x^{(k)}).\end{cases}\tag{2.38}$$

However, as we may see, (2.38) is identical with that of Bai et al.'s [2] as is given in (2.8) provided that the roles of the parameters ω, τ, γ in (2.8) are played by $\frac{\omega}{\alpha+\gamma}$, $\frac{2\omega}{2-\gamma}$, $\frac{2\gamma}{2-\gamma}$ in (2.38), respectively. Hence, if we put *accents* to the three parameter s of the present MASOR method to distinguish them from those of Bai et al.'s [2], then from (2.38) and (2.8)–(2.11), we will have

$$\frac{\omega'}{\alpha'+\gamma'}\in(0,2),\quad \frac{2\omega'}{2-\gamma'}\in\left(0,\frac{4}{\omega\mu_{\max}}\right),\quad \frac{2\gamma'}{2-\gamma'}\in\left(\tau-\frac{1}{\mu_{\max}},\frac{\tau}{2}+\frac{2-\omega}{\omega\mu_{\max}}\right).\tag{2.39}$$

Finally, to find the optimal parameters, we set from (2.20) $\tau_{opt}=\gamma_{opt}=\frac{1}{\sqrt{\mu_{\max}\mu_{\min}}}$, $\omega_{opt}=\frac{4\sqrt{\mu_{\max}\mu_{\min}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2}$ and from the correspondence between the parameters in (2.39) and (2.20) we have

$$\frac{2\omega'_{opt}}{2-\gamma'_{opt}}=\frac{2\gamma'_{opt}}{2-\gamma'_{opt}}=\frac{1}{\sqrt{\mu_{\max}\mu_{\min}}},\quad \frac{\omega'_{opt}}{\alpha'_{opt}+\gamma'_{opt}}=\frac{4\sqrt{\mu_{\max}\mu_{\min}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2}.\tag{2.40}$$

from which we obtain

$$\omega'_{opt} = \gamma'_{opt} = \frac{2}{1+2\sqrt{\mu_{\max}\mu_{\min}}}, \quad \alpha'_{opt} = \frac{\left(\sqrt{\mu_{\max}} - \sqrt{\mu_{\min}}\right)^2}{2\sqrt{\mu_{\max}\mu_{\min}}\left(1+2\sqrt{\mu_{\max}\mu_{\min}}\right)}. \tag{2.41}$$

Remark 2.8 It is noted that the expressions for the optimal parameters ω'_{opt} and α'_{opt} in (2.41) for the analogous optimal two-parameter iterative method were found in [11].

Finally, based on all of the above, we can obtain from Theorems 1, 2, and 3 of [9] in a condense form after some simple operations.

Theorem 2.7 *(Condensed form of extension of Theorems 1,2,3 of [9]) Under the notation, assumptions, and restrictions so far, if μ is an eigenvalue of $\mathcal{J} = Q^{-1}B^TA^{-1}B$ ($\mu \in \sigma(\mathcal{J})$, then $\lambda \in \sigma\left(\mathcal{T}(\alpha', \omega', \gamma')\right)$ then either $\lambda = 1 - \frac{\omega'}{\alpha'+\gamma}$ or λ is a root of the quadratic equation*

$$\lambda^2 - \left(2 - \frac{\omega'}{\alpha'+\gamma'} - \frac{2\omega'\gamma'}{(\alpha'+\gamma')(2-\gamma')}\mu\right)\lambda + (1 - \frac{\omega'}{\alpha'+\gamma'}) + \frac{2\omega'(\omega'-\gamma')}{(\alpha'+\gamma')(2-\gamma')}\mu = 0. \tag{2.42}$$

In view of (2.39)–(2.42) and the second part of Theorem 2.3, the optimal spectral radius of the MASOR iterative method is given by

$$\rho\left(\mathcal{T}(\alpha'_{opt}, \omega'_{opt}, \gamma'_{opt})\right) = \frac{\sqrt{\mu_{\max}} - \sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}} + \sqrt{\mu_{\min}}}. \tag{2.43}$$

3 Equivalence of nonsingular symmetric three-parameter iterative methods

In this section, we will summarize the results of all the four methods of Section 2, 2.1, 2.2, and 2.4 and the technique for the APIU iterative method so that the equivalence among the parameters involved, their intervals of convergence, their optimal parameters, and the coincidence of the optimal spectral radii of the iteration matrices of the methods will become much clearer. It should be pointed out that to make things simpler we will restrict to the case $\sigma(\mathcal{J}) \subset [\mu_{\min}, \mu_{\max}] \subset (0, +\infty)$, $m > n$, $\omega \in (0, 2)$, $\tau > 0$, and $\gamma \in \left(\tau - \frac{1}{\mu_{\max}}, \frac{\tau}{2} + \frac{2-\omega}{\omega\mu_{\max}}\right)$ only. Then, the equivalence of the aforementioned four iterative methods will be established.
(Note that it is understood that the more general case would be that we should take the technique for APIU iterative method by Huang and Wang [18], extend all the other four aforementioned methods, and prove their equivalence but this would make the paper too long and is, in our opinion, straightforward by using Theorems 2.5 and 2.6.)

In the above four methods, some of the expressions may be slightly modified so that the aforementioned equivalence to the basic APIU iterative method by Bai et al. [2] will be brought to a one-to-one correspondence; in addition, some of their parameters will be primed because they may define different quantities from the ones of the basic APIU of the previously mentioned methods.

The equivalence of the four methods will be easily shown based on the technique of APIU by Huang and Wang [18] and Theorem 2.5. This is omitted for the reasons explained above.

General considerations .

Problem (1.1) with its associated entities and their properties.

$Q \approx B^T A^{-1} B$ symmetric positive definite,

$$\mathcal{J} := Q^{-1} B^T A^{-1} B, \mu \in \sigma(\mathcal{J}) \subset [\mu_{min}, \mu_{\max}] \subset (0, +\infty).$$

APIU iterative method by Bai, Z.-Z.-Parlett, B.N.-Wang Z.-Q. [2]**:** Iterative parameters: $\omega \quad \tau, \ \gamma.$

General form of iterative algorithm (2.8):

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega A^{-1}(p - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \tau Q^{-1}(B^T x^{(k)} - q) + \gamma Q^{-1} B^T (x^{(k+1)} - x^{(k)}). \end{cases}$$

The eigenvalues of the iteration matrix $\mathcal{T}$ of algorithm (2.8) are $\lambda = 1 - \omega$, and all others are given by the roots of the functional (2.10):

$$\lambda^2 - (2 - \omega - \omega\gamma\mu)\lambda + (1-\omega) + \omega(\tau - \gamma)\mu = 0.$$

Intervals of convergence for the parameters involved (2.11):

$$\omega \in (0, \ 2), \quad \tau \in \left(0, \frac{4}{\omega\mu_{\max}}\right), \quad \gamma \in \left(\tau - \frac{1}{\mu_{\max}}, \ \frac{\tau}{2} + \frac{2-\omega}{\omega\mu_{\max}}\right).$$

Optimal parameters (2.20): $\omega_{opt} = \frac{4\sqrt{\mu_{\max}\mu_{\min}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2}$, $\tau_{opt} = \gamma_{opt} = \frac{1}{\sqrt{\mu_{\max}\mu_{\min}}}$.

Optimal spectral radius of (2.8): $\rho\left(\mathcal{T}(\omega_{opt}, \tau_{opt}, \gamma_{opt}\right) = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}$.

GMESOR iterative method by Louka, M.A.-Missirlis, N.M. [20] **(Louka, M.** [19]**):** Iterative parameters: $\tau_1 \quad \tau_2, \ a, \ \omega_2.$
General form of iterative algorithm (2.16):

$$\begin{cases} x^{(k+1)} = (1-\tau_1)x^{(k)} + \tau_1 A^{-1}(p - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \frac{\tau_2}{1-a\omega_2} Q^{-1}(B^T x^{(k)} - q) + \frac{\omega_2}{1-a\omega_2} Q^{-1} B^T (x^{(k+1)} - x^{(k)}). \end{cases}$$

The eigenvalues of the iteration matrix $\mathcal{T}$ of algorithm (2.16) are $\lambda = 1 - \tau_1$, and all the others are the roots of the functional (2.18):

$$\lambda^2 - (2 - \tau_1 - \tau_1 \frac{\omega_2}{1-a\omega_2}\mu)\lambda + (1-\tau_1) + \tau_1\left(\frac{\tau_2}{1-a\omega_2} - \frac{\omega_2}{1-a\omega_2}\right)\mu = 0.$$

A one-to-one correspondence of the parameters of the GMESOR and APIU iterative methods:

$$\tau_1 = \omega, \quad \frac{\omega_2}{1-a\omega_2} = \gamma, \quad \frac{\tau_2}{1-a\omega_2} = \tau.$$

Note: As is seen, γ and τ of APIU were given as functions of two parameters one of which ($a \neq \frac{1}{\omega_2}$) is redundant. For $a = 0$, GMESOR $\equiv$ APIU.

Optimal parameters and optimal spectral radius in terms of a (2.19):

$$\omega_{2_{opt}} = \tau_{2_{opt}} = \frac{1}{a+\sqrt{\mu_{\max}\mu_{\min}}}, \quad a \neq -\sqrt{\mu_{\max}\mu_{\min}}, \quad \tau_{1_{opt}} = \frac{4\sqrt{\mu_{\max}\mu_{\min}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2},$$
$$\rho\left(\mathcal{T}(\tau_{1_{opt}}, \tau_{2_{opt}}, \omega_{2_{opt}}, a)\right) = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}.$$

GMPSD iterative method by Louka, M.A.-Missirlis, N.M. [20] **(Louka, M.** [19]**):** Iterative parameters: $\tau_1 \quad \tau_2, \quad a, \quad \omega_1, \quad \omega_2.$

General form of iterative algorithm (2.21):

$$\begin{cases} y^{(k+1)} &= y^{(k)} + \frac{1}{(1-a\omega_2)[1-(1-a)\omega_2]} Q^{-1} \left\{B^T \left[(\tau_2 - \tau_1\omega_2)x^{(k)} \right.\right. \\ &+ \left.\left. \tau_1\omega_2 A^{-1}(p - By^{(k)})\right] - \tau_2 q\right\}, \\ x^{(k+1)} &= (1-\tau_1)x^{(k)} + A^{-1}\{B[(\omega_1 - \tau_1)y^{(k)} - \omega_1 y^{(k+1)}] + \tau_1 p\}. \end{cases}$$

The eigenvalues of the iteration matrix $\mathcal{T}$ of algorithm (2.21) are $\lambda = 1 - \tau_1$, and all the others are the roots of the functional (2.22):

$$\lambda^2 - (2 - \tau_1 - \tau_1 \frac{\left(\omega_2 + \frac{\tau_2}{\tau_1}\omega_1 - \omega_1\omega_2\right)}{(1-a\omega_2)[1-(1-a)\omega_2]}\mu)\lambda + (1-\tau_1)$$
$$+\tau_1\left(\frac{\tau_2}{(1-a\omega_2)[1-(1-a)\omega_2]} - \frac{\omega_2 + \frac{\tau_2}{\tau_1}\omega_1 - \omega_1\omega_2}{(1-a\omega_2)[1-(1-a)\omega_2]}\right)\mu = 0.$$

A one-to-one correspondence of the parameters of the GMPSD and APIU iterative methods:

$$\tau_1 = \omega, \quad \frac{\omega_2 + \frac{\tau_2}{\tau_1}\omega_1 - \omega_1\omega_2}{(1-a\omega_2)[1-(1-a)\omega_2]} = \gamma, \quad \frac{\tau_2}{(1-a\omega_2)[1-(1-a)\omega_2]} = \tau.$$

Note: As is seen, γ and τ in APIU were given as functions of three parameters two of which (a and ω_2) are redundant. For $\omega_2 = 0$, GMPSD $\equiv$ APIU.

Optimal parameters and optimal spectral radius in terms of a and ω_2 (2.23):

$$\tau_{1_{opt}} = \frac{4\sqrt{\mu_{\max}\mu_{\min}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2}, \quad \tau_{2_{opt}} = \frac{(1-a\omega_2)[1-(1-a)\omega_2]}{\sqrt{\mu_{\max}\mu_{\min}}},$$
$$\omega_{1_{opt}} = \frac{\tau_{1_{opt}}(\tau_{2_{opt}}-\omega_2)}{\tau_{2_{opt}}-\tau_{1_{opt}}\omega_2},$$
$$\rho\left(\mathcal{T}(\tau_{1_{opt}}, \tau_{2_{opt}}, \omega_{1_{opt}}, a, \omega_2)\right) = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}.$$

MASOR iterative method by Feng, T.-T.-Guo, X.-P.-Chen, G.-L. [9]**:** Iterative parameters: $\omega' \quad \alpha', \quad \gamma'.$

General form of iterative algorithm (2.38):

$$\begin{cases} x^{(k+1)} = (1 - \frac{\omega'}{\alpha'+\gamma'})x^{(k)} + \frac{\omega'}{\alpha'+\gamma'} A^{-1}(p - By^{(k)}). \\ y^{(k+1)} = y^{(k)} + \frac{2\omega'}{2-\gamma'} Q^{-1}(B^T x^{(k)} - q) + \frac{2\gamma'}{2-\gamma'} Q^{-1} B^T (x^{(k+1)} - x^{(k)}). \end{cases}$$

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The eigenvalues of the iteration matrix $\mathcal{T}$ of algorithm (2.38) are $\lambda = 1 - \frac{\omega'}{\alpha'+\gamma'}$, and all the others are given by the roots of the functional (2.42):

$$\lambda^2 - \left(2 - \frac{\omega'}{\alpha'+\gamma'} - \frac{\omega'}{\alpha'+\gamma'}\frac{2\gamma'}{2-\gamma'}\mu\right)\lambda + \left(1 - \frac{\omega'}{\alpha'+\gamma'}\right) + \frac{\omega'}{\alpha'+\gamma'}\left(\frac{2\omega'}{2-\gamma'} - \frac{2\gamma'}{2-\gamma'}\right)\mu = 0.$$

A one-to-one correspondence of the parameters of the MASOR and APIU iterative methods from (2.39):

$$\frac{\omega'}{\alpha'+\gamma'} = \omega, \quad \frac{2\omega'}{2-\gamma'} = \tau, \quad \frac{2\gamma'}{2-\gamma'} = \gamma.$$

Optimal parameters from (2.41):

$$\omega'_{opt} = \gamma'_{opt} = \frac{2}{1+2\sqrt{\mu_{\max}\mu_{\min}}}, \quad \alpha'_{opt} = \frac{\left(\sqrt{\mu_{\max}} - \sqrt{\mu_{\min}}\right)^2}{2\sqrt{\mu_{\max}\mu_{\min}}\left(1+2\sqrt{\mu_{\max}\mu_{\min}}\right)}$$

Optimal spectral radius from (2.43):

$$\rho\left(\mathcal{T}(\alpha'_{opt}, \omega'_{opt}, \gamma'_{opt})\right) = \frac{\sqrt{\mu_{\max}} - \sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}} + \sqrt{\mu_{\min}}}.$$

4 Singular symmetric three-parameter iterative methods

In the previous section, it was proved that all five three-parameter iterative methods (it is reminded that there were two of them by Louka and Missirlis [20]) proposed for the solution of the nonsingular symmetric saddle-point problem (1.1) are equivalent. (Note that for the time being we are leaving out the additional issues of Remarks 2.5 and 2.6 of Huang and Wang's APIU method.)

In this section, we show the equivalence of the above five methods by considering as their representative the Bai et al.'s *APIU* iterative method when the singular symmetric saddle-point problem has the same form as in (1.1) except that $m \geq n$, the matrix B is rank deficient with $\text{rank}(B) = r < n$ and the system is consistent, i.e., $[p^T \; -q^T]^T \in \text{range}(\mathcal{A})$.

Let the matrix B have the following *singular value decomposition* (SVD) form (see Horn and Johnson [17])

$$U^T BV = \begin{bmatrix} \Sigma_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} =: S, \quad S^T = V^T B^T U = \begin{bmatrix} \Sigma_r & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r}, \end{bmatrix}, \tag{4.1}$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and $\Sigma_r = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ being the singular values of B.

In this case, the iteration matrix $\mathcal{T}(\omega, \tau, \gamma)$ in (2.6) becomes similar to

$$
\begin{aligned}
\widehat{\mathcal{T}} &= \mathrm{diag}(U^T, V^T)\mathcal{T}(\omega, \tau, \gamma)\mathrm{diag}(U, V) \\
&= \begin{bmatrix} (1-\omega)I_m & -\omega U^T A^{-1} B V \\ (\tau - \omega\gamma) V^T Q^{-1} B^T U & I_n - \omega\gamma V^T Q^{-1} B^T A^{-1} B V \end{bmatrix} \\
&= \begin{bmatrix} (1-\omega)I_m & -\omega U^T A^{-1} U U^T B V \\ (\tau - \omega\gamma) V^T Q^{-1} V V^T B^T U & I_n - \omega\gamma V^T Q^{-1} V V^T B^T U U^T A^{-1} U U^T B V \end{bmatrix} \\
&= \begin{bmatrix} (1-\omega)I_m & -\omega \widehat{A}^{-1} S \\ (\tau - \omega\gamma) \widehat{Q}^{-1} S^T & I_n - \omega\gamma \widehat{Q}^{-1} S^T \widehat{A}^{-1} S \end{bmatrix},
\end{aligned}
\tag{4.2}
$$

where $\widehat{A}^{-1} = U^T A^{-1} U$, $\widehat{Q}^{-1} = V^T Q V$ and $\widehat{A}$, $\widehat{Q}$ are orthogonally similar to A, Q, respectively.

Before we prove the main theorem of this section, which applies to all five three-parameter iterative methods, we present a number of statements.

Lemma 4.1 *(Definition (4.8) and Exercise (4.9) on p. 152 of Berman and Plemmons [5]): Let $\mathcal{T} \in \mathbb{R}^{s\times s}$. Then, $\mathcal{T}$ is semi-convergent if and only if each of the following conditions holds:*

1. $\rho(\mathcal{T}) \le 1$.
2. *If* $\rho(\mathcal{T}) = 1$ *then* $\mathit{index}(I_s - \mathcal{T}) = 1$ $\left(\mathit{index}(I_s - \mathcal{T}) = 1 \Leftrightarrow \mathit{rank}\left((I_s - \mathcal{T})^2\right) = \mathit{rank}(I_s - \mathcal{T})\right)$.
3. *If* $\rho(\mathcal{T}) = 1$ *then* $\lambda \in \sigma(\mathcal{T})$ *with* $|\lambda| = 1$ *implies* $\lambda = 1$.

A lemma equivalent to Lemma 4.1 is the following.

Lemma 4.2 *(Lemma 2.2 of [32]) Let $\mathcal{H} \in \mathbb{C}^{l\times l}$ and $I_{s-l} \in \mathbb{C}^{(s-l)\times(s-l)}$ be the identity matrix, then the block partitioned matrix*

$$
\mathcal{T} = \begin{bmatrix} \mathcal{H} & 0_{l,s-l} \\ \mathcal{L} & I_{s-l} \end{bmatrix}
\tag{4.3}
$$

is semi-convergent if and only if either $\mathcal{L} = 0$ and $\mathcal{H}$ is semi-convergent or $\rho(\mathcal{H}) < 1$.

Definition 4.1 If $\mathcal{T}$ of Lemmas 4.1 and 4.2 is semi-convergent, then the quantity

$$
\gamma(\mathcal{T}) = \max\{|\lambda| \big| \lambda \in \sigma(\mathcal{T}), \lambda \neq 1\}
\tag{4.4}
$$

is called "semi-convergence factor" and plays the role of the spectral radius of a convergent $\mathcal{T}$.

Lemma 4.3 *Let $\mathcal{T} \in \mathbb{R}^{s\times s}$ be semi-convergent. Then, the iterative scheme $z^{(k+1)} = \mathcal{T}\ddagger^{(k)} + c$, $k = 0, 1, 2, \cdots$, $z^{(0)} \in \mathbb{R}^s$, semi-converges, namely*

$$
\lim_{k\to\infty} z^{(k)} = (I_s - \mathcal{T})^D c + (I_s - E) z^{(0)}, \quad E = (I_s - \mathcal{T})(I_s - \mathcal{T})^D,
\tag{4.5}
$$

(see Berman and Plemmons [5], formula (6.14) on p. 199, where $(\cdot)^D$ denotes Drazin inverse (see same reference Definition 4.10 on p. 118)).

Theorem 4.1 *Let the singular symmetric saddle-point problem (1.1) where rank$(B) = r < n \leq m$ and $[p^T \; -q^T]^T \in$ range$(\mathcal{A})$. Then, for any $z^{(0)} = \left[x^{(0)^T} \; y^{(0)^T}\right]^T \in \mathbb{R}^{m+n}$ the APIU iterative method (2.5) semi-converges to a solution of the singular system (1.1) for any triad (ω, τ, γ) satisfying the conditions in (2.11), where $\mu_{\max}$ is the largest eigenvalue of $\mathcal{J} = Q^{-1}B^TA^{-1}B$, with $Q \in \mathbb{R}^{n\times n}$ being symmetric positive definite and $\mu_{\min}$ the smallest positive eigenvalue of $\mathcal{J}$. The same holds true for all other three-parameter*

iterative methods presented in Section 2 *provided their parameters are interpreted the right way.*

Proof We follow a way of proof based on Lemma 4.2 and not the one based on Lemma 4.1 as this was done in [11]. First, we partition U, V, A^{-1}, and Q^{-1} into four blocks each so that their (1, 1) blocks are $r \times r$ matrices. Hence, we have that

$$\widehat{A}^{-1} = \begin{bmatrix} U_1^T(A^{-1})_{11}U_1 & U_1^T(A^{-1})_{12}U_1 \\ U_2^T(A^{-1})_{21}U_1 & U_2^T(A^{-1})_{22}U_2 \end{bmatrix} = \begin{bmatrix} (\widehat{A}^{-1})_{11} & (\widehat{A}^{-1})_{12} \\ (\widehat{A}^{-1})_{21} & (\widehat{A}^{-1})_{22} \end{bmatrix},$$

$$\widehat{Q}^{-1} = \begin{bmatrix} V_1^T(Q^{-1})_{11}V_1 & V_1^T(Q^{-1})_{12}V_1 \\ V_2^T(Q^{-1})_{21}V_1 & V_2^T(Q^{-1})_{22}V_2 \end{bmatrix} = \begin{bmatrix} (\widehat{Q}^{-1})_{11} & (\widehat{Q}^{-1})_{12} \\ (\widehat{Q}^{-1})_{21} & (\widehat{Q}^{-1})_{22} \end{bmatrix}.$$

Then, $\widehat{\mathcal{T}}$ in (4.2) becomes

$$\widehat{\mathcal{T}} = \left[\begin{array}{ccc|c} (1-\omega)I_r & 0_{r,m-r} & -\omega(\widehat{A}^{-1})_{11}\Sigma & 0_{r,n-r} \\ 0_{m-r,r} & (1-\omega)I_{m-r} & -\omega(\widehat{A}^{-1})_{21}\Sigma & 0_{m-r,n-r} \\ (\tau-\omega\gamma)(\widehat{Q}^{-1})_{11}\Sigma & 0_{r,m-r} & I_r - \omega\gamma(\widehat{Q}^{-1})_{11}\Sigma(\widehat{A}^{-1})_{11}\Sigma & 0_{r,n-r} \\ \hline (\tau-\omega\gamma)(\widehat{Q}^{-1})_{21}\Sigma & 0_{n-r,m-r} & -\omega\tau(\widehat{Q}^{-1})_{21}\Sigma(\widehat{A}^{-1})_{21}\Sigma & I_{n-r} \end{array}\right].$$

Obviously, $\widehat{\mathcal{T}}$ has the form

$$\widehat{\mathcal{T}} = \begin{bmatrix} \widehat{\mathcal{H}} & 0_{m+r,n-r} \\ \widehat{\mathcal{L}} & I_{n-r} \end{bmatrix}, \tag{4.6}$$

where $\rho(\widehat{\mathcal{H}}) < 1$, with the values of the parameters of $\widehat{\mathcal{H}}$ being used in (2.10) are in the intervals defined in (2.11) and the optimal parameters are given by the expressions in (2.20). Note that $\mu_{\min}$ and $\mu_{\max}$ are the smallest and the largest positive eigenvalues of $\widehat{\mathcal{H}}$ and the optimal semi-convergence factor of the matrix $\mathcal{T}$ is given by

$$\gamma(\mathcal{T}_{\omega_{opt},\tau_{opt},\gamma_{opt}}) = \gamma(\widehat{\mathcal{T}}_{\omega_{opt},\tau_{opt},\gamma_{opt}}) = \rho(\widehat{\mathcal{H}}_{\omega_{opt},\tau_{opt},\gamma_{opt}}) = \frac{\sqrt{\mu_{\max}} - \sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}} + \sqrt{\mu_{\min}}} < 1. \tag{4.7}$$

This effectively proves that the matrix $\widehat{\mathcal{T}}$ and its similar $\mathcal{T}$ are semi-convergent and so are the other four equivalent to them three-parameter iterative methods presented in Section 2. □

5 Numerical examples

To the best of our knowledge, the authors Zheng-Bai-Yang were the first to theoretically work out the singular symmetric saddle-point problem and presented two

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numerical examples 5.1 and 5.2 in [32]. Example 5.1, restricted to its nonsingular symmetric part A and B, with $\mathrm{rank}(B) = n$, is Example 5.1 of [2] taken, in turn, from [1]. The same nonsingular symmetric example was also used in [8, 14, 25, 27, 33] and in many others. Technical modifications of B to make $\mathcal{A}$ singular were first appeared in [32], as in Examples 5.1, 5.2, subsequently in [7, 15, 21–23, 29, 30, 34, 35, 37, 38] and maybe in others. The authors of [32] kept the matrix A of Example 5.1 of [2] and artificially constructed the matrices B to make $\mathcal{A}$ singular. We preferred to use Example 5.2 rather than 5.1 since in the former much more information was given than the latter in [32] and so we can use it for comparison purposes.

The matrix blocks A and B of $\mathcal{A}$ of (1.1) are as follows:

$$\begin{gathered} A = \begin{bmatrix} I_l \otimes T + T \otimes I_l & 0 \\ 0 & I_l \otimes T + T \otimes I_l \end{bmatrix} \in \mathbb{R}^{2l^2 \times 2l^2}, \quad l \ \text{ even}, \\ T = \frac{1}{h^2}\mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{l \times l}, \quad F = \frac{1}{h}\mathrm{tridiag}(-1, 1, 0) \in \mathbb{R}^{l \times l}, \quad h = \frac{1}{l+1}, \\ B = \widehat{B}\widetilde{B} \in \mathbb{R}^{2l^2 \times l^2}, \quad \widehat{B} = \begin{bmatrix} I_l \otimes F \\ F \otimes I_l \end{bmatrix} \in \mathbb{R}^{2l^2 \times l^2}, \quad \widetilde{B} = I \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \in \mathbb{R}^{l^2 \times l^2}, \end{gathered} \tag{5.1}$$

where h is the discretization mesh size. Obviously, $m = 2l^2$ and $n = l^2$. Hence, the total number of components of the vectors involved is $m + n = 3l^2$.

Four expressions for the preconditioning matrix Q, as an approximation to the matrix $B^T A^{-1} B$, were chosen as is indicated in Table 1. These expressions were previously used in the parameterized Uzawa (PU) method [32].

All numerical experiments were implemented in MATLAB (version 8.2.0.701 (R2013b), on a personal computer with machine precision 10^{-16}, 3.50 GHz central processing unit (Intel(R) Core(TM)i3), 4G memory and Windows 10 operating system. For the APIU method, all numerical examples were started with an initial vector $\left[x^{(0)T} \ y^{(0)T}\right]^T$ and terminated when the current iteration satisfied $\mathrm{ERR} \le \varepsilon$, where ε is a small positive number, or when a prescribed maximum iteration number was exceeded. ERR denotes the ratio of the norm of the residual of the iteration vector at hand RES over that of the initial vector. Both ERR and RES are defined by

$$\mathrm{ERR} := \frac{\sqrt{\|p - Ax^{(k)} - By^{(k)}\|_2^2 + \|q - B^T x^{(k)}\|_2^2}}{\sqrt{\|p - Ax^{(0)} - By^{(0)}\|_2^2 + \|q - B^T x^{(0)}\|_2^2}} \le \varepsilon. \tag{5.2}$$

Table 1 Choices of the matrix Q

Case	Matrix Q	Description
I	$\widehat{B}^T \widehat{A}^{-1} \widehat{B}$	$\widehat{A} = \mathrm{tridiag}(A)$
II	$\widehat{B}^T \widehat{A}^{-1} \widehat{B}$	$\widehat{A} = \mathrm{diag}(A)$
III	$\mathrm{tridiag}(\widehat{B}^T \widehat{A}^{-1} \widehat{B})$	$\widehat{A} = \mathrm{tridiag}(A)$
IV	$\mathrm{tridiag}(\widehat{B}^T \widehat{A}^{-1} \widehat{B})$	$\widehat{A} = A$

Note that if and only if the initial vector $\left[x^{(0)^T}\ y^{(0)^T}\right]^T$ is the zero vector then the relation for the ERR is simplified to

$$\mathrm{ERR} := \frac{\sqrt{\|p - Ax^{(k)} - By^{(k)}\|_2^2 + \|q - B^T x^{(k)}\|_2^2}}{\sqrt{\|p\|_2^2 + \|q\|_2^2}} \leq \varepsilon.\ ) \tag{5.3}$$

The norm of the residual vector RES is given by

$$\mathrm{RES} = \sqrt{||p - x^{(k)} - By^{(k)}||_2^2 + ||q - B^T x^{(k)}||_2^2}. \tag{5.4}$$

In the examples, we ran $\varepsilon = 10^{-6}$ was taken.

The right hand side vector $\left[p^T\ \ -q^T\right]^T \in \mathbf{R}^{m+n}$ was chosen such that the exact solution of the augmented linear system (1.1) is $\left[x_*^T\ y_*^T\right]^T = [1\,1\,\cdots\,1]^T \in \mathbf{R}^{m+n}$. Note that the vector $\left[x_{**}^T\ y_{**}^T\right]^T \in \mathbb{R}^{m+n}$, with sub-vectors $x_{**} = [1\,1\,\cdots\,1]^T \in \mathbb{R}^m$, $y_{**} = [0\,0\,\cdots\,0]^T \in \mathbb{R}^n$, constitutes also an obvious solution.

In Table 2 and for the Case I only, we present $\mu_{\min}$ and $\mu_{\max}$ as well as the optimal values ω_{opt} and $\tau_{opt} = \gamma_{opt}$ for selected values of l ($m = 2l^2$, $n = l^2$) which were considered in [32]. The optimal values ω_{opt}, $\tau_{opt} = \gamma_{opt}$ for the cases II–IV and for the same values of l will be given in Tables 4, 5, and6.

In the following four Tables 3, 4, 5, and 6, the results obtained are depicted when Example 5.1 was worked out with the indicated sizes for m and n for all four choices of the matrix Q of Table 1 (cases I–IV) and with three different initial vectors. The sizes m and n, the two optimal parameters ω_{opt} and τ_{opt} ($= \gamma_{opt}$), the iteration numbers (IT), the CPU times in seconds (CPU), and the residuals (RES) of the APIU iterative method can be seen in them. (Note that it should be said that (i) $\mu_{\min}$ and $\mu_{\max}$ had also been found for the last three choices of Q but we thought it was not necessary to give them here and (ii) all $\mu_{\min}$ and $\mu_{\max}$ in our experiments were found using the corresponding MATLAB function with a tolerance of 10^{-12} or less.)

If we look at the CPU times in all four Tables 3–6, we see that there are small differences regarding them depending on the choice of the initial vectors $[x^{(0)^T}\ y^{(0)^T}]^T$.

Besides the four Tables 3– 6 and the optimal results just presented using the APIU iterative method, we also depict in Table 7 the corresponding results when using the MINRES and the preconditioned MINRES (PMINRES) iterative methods; the latter with the same choices for the matrix Q. Since by default the two Krylov subspace methods use the zero vector as the starting vector the relevant comparisons should

Table 2 **Case I.** $\sigma(Q)\backslash\{0\} \subset [\mu_{\min}, \mu_{\max}] \subset (0, +\infty)$

	$\mu_{\min}$	$\mu_{\max}$	ω_{opt}	$\tau_{opt} = \gamma_{opt}$
$l = 8$	2.7555	7.4933	0.9400	0.2201
$l = 16$	2.6918	7.8577	0.9316	0.2174
$l = 24$	2.6783	7.9352	0.9298	0.2169
$l = 32$	2.6734	7.9633	0.9291	0.2167

Table 3 Case I

	$m = 128$	$m = 512$	$m = 1152$	$m = 2048$
	$n = 64$	$n = 256$	$n = 576$	$n = 1024$
ω_{opt}	0.9400	0.9316	0.9298	0.9291
$\tau_{opt} = \gamma_{opt}$	0.2201	0.2174	0.2169	0.2167
	IT = 10	IT = 11	IT = 11	IT = 11
$x^{(0)} = [0\,0 \ldots 0]^T$	CPU = 0.0005	CPU = 0.0156	CPU = 0.0708	CPU = 0.2025
$y^{(0)} = [0\,0 \ldots 0]^T$	ERR = 8.7523e−07	ERR = 5.5615e−07	ERR = 7.1339e−07	ERR = 7.9475e−07
	RES = 6.3488e−04	RES = 1.9295e−03	RES = 6.4316e−03	RES = 1.4275e−02
	IT = 10	IT = 11	IT = 11	IT = 11
$x^{(0)} = [0\,0 \ldots 0]^T$	CPU = 0.0005	CPU = 0.0156	CPU = 0.0686	CPU = 0.2032
$y^{(0)} = [1\,1 \ldots 1]^T$	ERR = 8.7523e−07	ERR = 5.5615e−07	ERR = 7.1339e−07	ERR = 7.9475e−07
	RES = 6.3488e−04	RES = 1.9295e−03	RES = 6.4316e−03	RES = 1.4275e−02
	IT = 10	IT = 11	IT = 11	IT = 11
$x^{(0)} = [1\,0 \ldots 1\,0]^T$	CPU = 0.0003	CPU = 0.0153	CPU = 0.0707	CPU = 0.2076
$y^{(0)} = [1\,0 \ldots 1\,0]^T$	ERR = 4.6019e−07	ERR = 4.2020e−07	ERR = 5.8515e−07	ERR = 6.7923e−07
	RES = 3.5854e−04	RES = 1.5180e−03	RES = 5.4266e−03	RES = 1.2468e−02
	IT = 10	IT = 11	IT = 11	IT = 11
$x^{(0)} = [1\,0 \ldots 1\,0]^T$	CPU = 0.0003	CPU = 0.0153	CPU = 0.0707	CPU = 0.2076
$y^{(0)} = [1\,0 \ldots 1\,0]^T$	ERR = 4.6019e−07	ERR = 4.2020e−07	ERR = 5.8515e−07	ERR = 6.7923e−07
	RES = 3.5854e−04	RES = 1.5180e−03	RES = 5.4266e−03	RES = 1.2468e−02

Table 4 Case II

	$m = 128$	$m = 512$	$m = 1152$	$m = 2048$
	$n = 64$	$n = 256$	$n = 576$	$n = 1024$
ω_{opt}	0.9058	0.8938	0.8912	0.8902
$\tau_{opt} = \gamma_{opt}$	0.2523	0.2504	0.2501	0.2501
	IT = 12	IT = 13	IT = 14	IT = 14
$x^{(0)} = [0\,0 \ldots 0]^T$	RES = 7.1166e−04	CPU = 0.0196	CPU = 0.0868	CPU = 0.2578
$y^{(0)} = [0\,0 \ldots 0]^T$	ERR = 9.8109e−07	ERR = 7.7224e−07	ERR = 3.7683e−07	ERR = 4.5747e−07
	RES = 7.1166e−04	RES = 2.6792e−03	RES = 3.3973e−03	RES = 8.2171e−03
	IT = 12	IT = 13	IT = 14	IT = 14
$x^{(0)} = [0\,0 \ldots 0]^T$	CPU = 0.0004	CPU = 0.0202	CPU = 0.0890	CPU = 0.2622
$y^{(0)} = [1\,1 \ldots 1]^T$	ERR = 9.8109e−07	ERR = 7.7224e−07	ERR = 3.7683e−07	ERR = 4.5747e−07
	RES = 7.1166e−04	RES = 2.6792e−03	RES = 3.3973e−03	RES = 8.2171e−03
	IT = 12	IT = 13	IT = 13 r	IT = 14
$x^{(0)} = [1\,0 \ldots 1\,0]^T$	CPU = 0.0004	CPU = 0.0175	CPU = 0.0820	CPU = 0.2581
$y^{(0)} = [1\,0 \ldots 1\,0]^T$	ERR = 6.8513e−07	ERR = 6.4961e−07	ERR = 9.7458e−07	ERR = 4.1522e−07
	RES = 5.3379e−04	RES = 2.3468e−03	RES = 9.0381e−03	RES = 7.6219e−03

Table 5 Case III

	$m = 128$	$m = 512$	$m = 1152$	$m = 2048$
	$n = 64$	$n = 256$	$n = 576$	$n = 1024$
ω_{opt}	0.9977	0.9975	0.9975	0.9975
$\tau_{opt} = \gamma_{opt}$	0.2400	0.2398	0.2397	0.2396
	IT = 4	IT = 4	IT = 4	IT = 4
$x^{(0)} = [0\,0\,...\,0]^T$	CPU = 0.0001	CPU = 0.0063	CPU = 0.0265	CPU = 0.0743
$y^{(0)} = [0\,0\,...\,0]^T$	ERR = 6.2547e−08	ERR = 6.9404e−08	ERR = 7.0589e−08	ERR = 7.0921e−08
	RES = 4.5370e−05	RES = 2.4079e−04	RES = 6.3640e−04	RES = 1.2739e−03
	IT = 4	IT = 4	IT = 4	IT = 4
$x^{(0)} = [0\,0\,...\,0]^T$	CPU = 0.0001	CPU = 0.0064	CPU = 0.0282	CPU = 0.0745
$y^{(0)} = [1\,1\,...\,1]^T$	ERR = 6.2547e−08	ERR = 6.9404e−08	ERR = 7.0589e−08	ERR = 7.0921e−08
	RES = 4.5370e−05	RES = 2.4079e−04	RES = 6.3640e−04	RES = 1.2739e−03
	IT = 4	IT = 4	IT = 4	IT = 4
$x^{(0)} = [1\,0\,...\,1\,0]^T$	CPU = 0.0002	CPU = 0.0058	CPU = 0.0257	CPU = 0.0768
$y^{(0)} = [1\,0\,...\,1\,0]^T$	ERR = 8.0689e−07	ERR = 7.5824e−07	ERR = 6.8342e−07	ERR = 6.1981e−07
	RES = 6.2866e−04	RES = 2.7392e−03	RES = 6.3379e−03	RES = 1.1377e−02

be made with the corresponding results of the Tables 3–6 for $x^{(0)} = 0 \in \mathbb{R}^m$ and $y^{(0)} = 0 \in \mathbb{R}^n$.

As a summary, regarding Tables 3–7, a number of points are made below which are pretty close to those given for Example 5.2 in [32].

Table 6 Case IV

	$m = 128$	$m = 512$	$m = 1152$	$m = 2048$
	$n = 64$	$n = 256$	$n = 576$	$n = 1024$
ω_{opt}	0.9990	0.9989	0.9989	0.9988
$\tau_{opt} = \gamma_{opt}$	0.2477	0.2489	0.2491	0.2492
	IT = 3	IT = 3	IT = 3	IT = 3
$x^{(0)} = [0\,0\,...\,0]^T$	CPU = 0.0002	CPU = 0.0052	CPU = 0.0206	CPU = 0.0628
$y^{(0)} = [0\,0\,...\,0]^T$	ERR = 3.5950e−07	ERR = 3.6667e−07	ERR = 3.9192e−07	ERR = 4.0261e−07
	RES = 2.6077e−04	RES = 1.2721e−03	RES = 3.5333e-03	RES = 7.2318e−03
	IT = 3	IT = 3	IT = 3	IT = 3
$x^{(0)} = [0\,0\,...\,0]^T$	CPU = 0.0001	CPU = 0.0049	CPU = 0.0207	CPU = 0.0679
$y^{(0)} = [1\,1\,...\,1]^T$	ERR = 3.5950e−07	ERR = 3.6667e−07	ERR = 3.9192e−07	ERR = 4.0261e−07
	RES = 2.6077e−04	RES = 1.2721e−03	RES = 3.5333e−03	RES = 7.2318e−03
	IT = 4	IT = 4	IT = 4	IT = 4
$x^{(0)} = [1\,0\,...\,1\,0]^T$	CPU = 0.0002	CPU = 0.0066	CPU = 0.0286	CPU = 0.0795
$y^{(0)} = [1\,0\,...\,1\,0]^T$	ERR = 2.6051e−07	ERR = 1.1926e−07	ERR = 1.1861e−07	ERR = 1.1626e−07
	RES = 2.0297e−04	RES = 4.3083e−04	RES = 1.1000e−03	RES = 2.1340e−03

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Table 7 MINRES and PMINRES

	$m = 128$ $n = 64$	$m = 512$ $n = 256$	$m = 1152$ $n = 576$	$m = 2048$ $n = 1024$
MINRES	IT = 54	IT = 99	IT = 125	IT = 158
	CPU = 0.0032	CPU = 0.0496	CPU = 0.2188	CPU = 0.7492
	ERR = 6.0502e−07	ERR = 8.2003e−07	ERR = 9.7611e−07	ERR = 8.9558e−07
	RES = 4.3887e−04	RES = 2.8449e−03	RES = 8.8001e−03	RES = 1.6087e−02
	IT = 53	IT = 91	IT = 115	IT = 140
PMINRES	CPU = 0.0697	CPU = 0.0437	CPU = 0.2232	CPU = 0.8407
Case I	ERR = 7.6345e−07	ERR = 9.8838e−07	ERR = 9.4842e−07	ERR = 9.9928e−07
	RES = 8.4286e−04	RES = 4.8655e−03	RES = 1.1676e−02	RES = 2.4053e−02
	IT = 77	IT = 146	IT = 187	IT = 235
PMINRES	CPU = 0.0046	CPU = 0.0662	CPU = 0.3423	CPU = 1.2378
Case II	ERR = 6.9940e−07	ERR = 8.8528e−07	ERR = 9.6244e−07	ERR = 9.5017e−07
	RES = 7.8387e−04	RES = 4.6438e−03	RES = 1.4195e−02	RES = 2.8379e−02
	IT = 62	IT = 107	IT = 135	IT = 164
PMINRES	CPU = 0.0040	CPU = 0.0516	CPU = 0.3754	CPU = 0.9511
Case III	ERR = 5.5294e−07	ERR = 8.2844e−07	ERR = 9.7470e−07	ERR = 9.1514e−07
	RES = 5.3880e−04	RES = 3.7478e−03	RES = 1.0279e−02	RES = 1.9705e−02
	IT = 10	IT = 10	IT = 9	IT = 9
PMINRES	CPU = 0.0020	CPU = 0.0100	CPU = 0.0426	CPU = 0.2011
Case IV	ERR = 4.1815e−07	ERR = 3.3008e−07	ERR = 8.5284e−07	ERR = 7.4259e−07
	RES = 5.4721e−04	RES = 2.1039e−03	RES = 1.1437e−02	RES = 2.0045e−02

1. The optimal cases I and II are pretty much equivalent as they are the optimal cases III and IV; the latter cases are far better than the former ones.
2. It seems that in the optimal cases III and IV, $\omega_{opt} \approx 1$ from below for all the values of l and, therefore, for the m and n considered.
3. The first three cases of PMINRES do not give better results than those of MINRES; also, case IV of PMINRES is superior to MINRES and to all the rest of PMINRES (cases I, II, III).
4. While the optimal cases I and II are pretty much equivalent to PMINRES (case IV), the optimal cases III and IV are obviously superior to the MINRES and PMINRES (cases I–IV).

To conclude the present section, we give one more table (Table 8), where convergence of the APIU iterative method is shown for various triads of the parameters (ω, τ, γ) chosen from their respective intervals of convergence. The extreme values of μ and those of $\omega_{opt}, \tau_{opt}, \gamma_{opt}$ are taken from the first row of the data of Table 2.

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Table 8 Convergence of APIU for various triads (ω, τ, γ)

$l = 8$ $(m = 128,\ n = 64)$			
$\omega_1 = 0.4700$	$\tau_1 = 0.1100$	$\gamma_1 = 0.0983$	IT = 44 CPU = 0.0013 ERR = 7.3254e−07 RES = 5.3137e−04
		$\gamma_2 = 0.3548$	IT = 42 CPU = 0.0013 ERR=8.1326e−07 RES=5.8993e−04
	$\tau_2 = 0.6779$	$\gamma_3 = 0.6208$	IT = 87 CPU = 0.0025 ERR=6.6236e−07 RES=4.8046e−04
		$\gamma_4 = 0.6971$	IT = 37 CPU = 0.0011 ERR = 6.6491e−07 RES = 4.8231e−04
$\omega_2 = 1.4700$	$\tau_3 = 0.1100$	$\gamma_5 = 0.0188$	IT = 35 CPU = 0.0010 ERR = 8.3354e−07 RES = 6.0463e−04
		$\gamma_6 = 0.0610$	IT = 18 CPU = 0.0009 ERR = 5.9152e−07 RES = 4.2908e−04
	$\tau_4 = 0.2916$	$\gamma_7 = 0.1701$	IT =160 CPU = 0.0078 ERR = 7.9131e−07 RES = 5.7400e−04
		$\gamma_8 = 0.1820$	IT =76 CPU = 0.0022 ERR = 7.1138e−07 RES = 5.1602e−04

To construct this table and at the same time have the triads (ω, τ, γ) as different as possible, we choose ω_1 and ω_2 as the midpoints of the intervals $(0, \omega_{opt})$ and $(\omega_{opt}, 2)$. Next, since $\tau \in (0, \frac{4}{\omega_i \mu_{\max}})$, $i = 1, 2$, we choose τ_1, τ_2 and τ_3, τ_4 using ω_1 and ω_2, respectively. Finally, the values of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and $\gamma_5, \gamma_6, \gamma_7, \gamma_8$ are chosen having in mind the four ranges for τ, based on ω_1 and ω_2, and the ranges for γ from (2.24).

6 Concluding remarks and discussion

1. Section 1 is an introductory section to the problems considered and solved in this work. A number of previous works that led us to consider the problems treated in this paper are cited.
2. In the introduction of Section 2, a brief reference is made to the *generalized inexact accelerated overrelaxation* (GIAOR) iterative method introduced by Bai, Parlett, and Wang in the beginning of Section 7 of [2] and specifically to a "simplified" version of it renamed later by Bai and Wang *accelerated parameterized inexact Uzawa* (APIU) iterative method. So, what we considered in the next subsections was the iterative solution of the nonsingular symmetric saddle-point problem using three parameters ω, τ, and γ, instead of the usual two ω and τ. The main seed and ideas of the method as well as the intervals of convergence of the three parameters can be found in the aforementioned Section 7 of [2].
3. In the rest and main part of Section 2, we considered five of the iterative schemes (maybe more have appeared in the literature) which we have come across. All of them are based directly or indirectly on the APIU iterative method [2]. We made a number of comments on them, we pointed out what their strong points are and made some improvements to the last but one method and completed the last one.
4. First, in Section 2.1, Bai et al. [2], in their pioneering work, proposed among others their three-parameter APIU iterative method and found intervals of convergence for all three parameters (ω, τ, γ) for the nonsingular symmetric saddle-point problem. This method is presented and their optimal parameters were given later after the equivalence between the APIU and GMESOR iterative methods was established.
5. Next, in Sections 2.2, 2.2.1, and 2.2.2, Louka and Missirlis [20] (see also [19]) proposed two iterative methods (GMESOR, GMPSD) and using a combination of analytical and geometrical tools succeeded in being the very first researchers who solved the three-parameter saddle-point problem completely. Surprisingly enough, in both methods, it was proved that $\tau_{opt} = \gamma_{opt}$, meaning that the optimal three-parameter iterative method was nothing but the well-known optimal two-parameter one which was solved by Bai et al. in [2]. The latter authors also found the regions of convergence parameters and the optimal parameters involved. The parameters a in [20] by Louka and Missirlis (see also [19]) in GMESOR as well as those of a, ω_2 in GMPSD three-parameter iterative methods were practically shown by the authors themselves that they were redundant.
6. Then, in Section 2.3, Huang and Wang [18] used the APIU iterative method and by purely analytical methods solved also completely the three-parameter saddle-point problem. As was pointed out in Remarks 2.5 and 2.6, Huang and Wang [18], besides the solution of the problem, as Louka and Missirlis did in [20], they also obtained for the first time in their analysis issues that had escaped the attention of all the previous researchers in the area. Specifically, (i) for the case $m = n$, different expressions for the convergence regions of

the three parameters involved from those of the case $m > n$ were obtained and (ii) regions of convergence of the three parameters when ω and τ take positive/negative or negative/positive values, respectively, were also obtained. The convergence regions of the parameters involved for $m > n$ and $m = n$ given in [18] were slightly modified. In Theorem 2.5, the modified ones were obtained and were presented in the expressions (2.25) and (2.26), respectively.

7. Finally, in Section 2.4, Feng et al. [9] presented their three-parameter MASOR iterative method for the solution of the nonsingular symmetric saddle-point problem but did not succeed in obtaining regions of convergence nor optimal parameters. What they missed regarding the previous two issues was completed by the present authors based mainly on the Louka and Missirlis's [20] and Huang and Wang's [18] works.
8. In Section 3, it is shown that all the four three-parameter iterative methods are equivalent for the solution of the nonsingular saddle-point problem. This is also true for their regions of convergence and their optimal parameters. A summary of all these issues is then briefly presented.
9. In Section 4, the singular symmetric saddle-point problem for the three-parameter iterative method was tackled and solved. As far as we know, this has been done for the very first time. The way we worked it out was based on the main Lemma 2.2 by Zheng et al. [32], instead of Lemma 3.4 of [11]. Naturally, the corresponding analysis was a little more complicated than that in [32]. Finally, it was proved that whatever holds for the regions of convergence and the optimal parameters of the nonsingular symmetric saddle-point problem does hold for the singular symmetric problem provided that we take out the zero eigenvalues from the spectrum of the matrix coefficient and work with a smaller nonsingular matrix (see text). In case, $m < n$ and $\text{rank}(B) = n' = m$, the optimal results and the ranges of convergence of the parameters involved found by Huang and Wang [18] for the special case $m = n$ with n' taking the place of n in the corresponding expressions can be applied.
10. In Section 5, we worked out Example 5.2 of [32] and any comments on it were given in the corresponding part of the text.

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