14.26

$$B=\int_0^{\frac{\pi}{4}}\left(1+\varepsilon\varphi^2x\right)dx=\int_0^{\frac{\pi}{4}}\left(\frac{1}{\sigma\upsilon\nu^2x}\right)dx=\int_0^{\frac{\pi}{4}}\left(\varepsilon\varphi x\right)'dx=\left[\varepsilon\varphi x\right]_0^{\frac{\pi}{4}}=1-0=1$$

$$\Delta=\int_0^{\frac{\pi}{4}}\left(\varepsilon\varphi^4x+\varepsilon\varphi^2x\right)dx=\int_0^{\frac{\pi}{4}}\varepsilon\varphi^2x\left(\varepsilon\varphi^2x+1\right)dx=\int_0^{\frac{\pi}{4}}\varepsilon\varphi^2x\cdot\left(\varepsilon\varphi x\right)'dx=\left[\frac{\varepsilon\varphi^3x}{3}\right]_0^{\frac{\pi}{4}}=\frac{1}{3}-0=\frac{1}{3}$$

14.27

$$A=\int_0^{\pi}e^{\eta\mu x}\sigma\upsilon\nu x dx=\left[e^{\eta\mu x}\right]_0^{\pi}=0-0=0$$

$$\Gamma=\int_0^{\pi}\eta\mu^4x\sigma\upsilon\nu x dx=\left[\frac{\eta\mu^5x}{5}\right]_0^{\pi}=0-0=0$$

$$\Delta=\int_1^2\frac{1}{x(x+1)}dx=\int_1^2\frac{(x+1)-x}{x(x+1)}dx=\int_1^2\left(\frac{(x+1)}{x(x+1)}-\frac{x}{x(x+1)}\right)dx=\int_1^2\left(\frac{1}{x}-\frac{1}{x+1}\right)dx=$$

$$=\left[\ln|x|-\ln|x+1|\right]_1^2=(\ln2-\ln3)-(0-\ln2)=2\ln2-\ln3=\ln4-\ln3=\ln\frac{4}{3}$$

$$E=\int_0^3\left|1-x^2\right|dx=\int_0^1\left(1-x^2\right)dx+\int_1^3\left(x^2-1\right)dx=\left[x-\frac{x^3}{3}\right]_0^1+\left[\frac{x^3}{3}-x\right]_1^3=$$

$$=\left(1-\frac{1}{3}\right)-0+(9-3)-\left(\frac{1}{3}-1\right)=\frac{22}{3}$$

$$Z=\int_0^{\pi}\sigma\upsilon\nu^3x dx=\int_0^{\pi}\sigma\upsilon\nu^2x\cdot\sigma\upsilon\nu x dx=\int_0^{\pi}\left(1-\eta\mu^2x\right)\cdot\sigma\upsilon\nu x dx=\int_0^{\pi}\sigma\upsilon\nu x dx-\int_0^{\pi}\eta\mu^2x\cdot\sigma\upsilon\nu x dx=$$

$$=\left[\eta\mu x\right]_0^{\pi}-\frac{1}{3}\left[\eta\mu^3x\right]_0^{\pi}=0-\frac{1}{3}\cdot0=0$$

14.29

$$A=\int_0^{\frac{\pi}{4}}\varepsilon\varphi^2x dx=\int_0^{\frac{\pi}{4}}\left(\frac{1}{\sigma\upsilon\nu^2x}-1\right)dx=\int_0^{\frac{\pi}{4}}\left(\varepsilon\varphi x\right)'dx-\int_0^{\frac{\pi}{4}}1dx=\left[\varepsilon\varphi x\right]_0^{\frac{\pi}{4}}-\left[x\right]_0^{\frac{\pi}{4}}=1-\frac{\pi}{4}$$

$$B=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sigma\varphi^2x dx=\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\left(\frac{1}{\eta\mu^2x}-1\right)dx=-\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\left(\sigma\varphi x\right)'dx-\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}1dx=-\left[\sigma\varphi x\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}-\left[x\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}=-1-\frac{\pi}{4}+\frac{\pi}{2}=-1+\frac{\pi}{4}$$

14.30

Αν η f έχει συνεχή δεύτερη παράγωγο στο [0,2] με f(0)=0 και f(2)=e^3 :

$$A=\int_0^2e^{2x}\left(f'(x)+2xf(x)\right)dx=\int_0^2e^{2x}f'(x)+e^{2x}\cdot2xf(x)dx=\int_0^2\left(e^{2x}f(x)\right)'dx=$$

$$\left[e^{2x}f(x)\right]_0^2=e^4f(2)-e^0f(0)=e^4\cdot e^3-1\cdot0=e^7$$

14.31

$$A=\int_0^1xe^{-x}dx=-\int_0^1x\left(e^{-x}\right)'dx=-\left(\left[xe^{-x}\right]_0^1-\int_0^1(x)'e^{-x}dx\right)=-\left(\left[xe^{-x}\right]_0^1+\int_0^1\left(e^{-x}\right)'dx\right)=$$

$$=-\left(\left[xe^{-x}\right]_0^1+\left[e^{-x}\right]_0^1\right)=-\left(\frac{1}{e}-0+\frac{1}{e}-1\right)=1-\frac{2}{e}$$

$$\mathrm{B}=\int_0^{\pi} x\sigma\upsilon\nu 3x dx=\frac{1}{3}\int_0^{\pi} x\left(\eta\mu 3x\right)' dx=\frac{1}{3}\left(\left[x\eta\mu 3x\right]_0^{\pi}-\int_0^{\pi}\left(x\right)'\eta\mu 3x dx\right)=$$
$$\frac{1}{3}\left(\left[x\eta\mu 3x\right]_0^{\pi}-\int_0^{\pi}\eta\mu 3x dx\right)=\frac{1}{3}\left(\left[x\eta\mu 3x\right]_0^{\pi}+\frac{1}{3}\int_0^{\pi}\left(\sigma\upsilon\nu 3x\right)' dx\right)=\frac{1}{3}\left(\left[x\eta\mu 3x\right]_0^{\pi}+\frac{1}{3}\left[\sigma\upsilon\nu 3x\right]_0^{\pi}\right)=$$
$$=\frac{1}{3}\left(0+\frac{1}{3}\left(-1-1\right)\right)=\frac{1}{3}\left(-\frac{2}{3}\right)=-\frac{2}{9}$$

$$\Gamma=\int_0^{\frac{\pi}{4}} x^2\eta\mu 2x dx=-\frac{1}{2}\int_0^{\frac{\pi}{4}} x^2\left(\sigma\upsilon\nu 2x\right)' dx=-\frac{1}{2}\left(\left[x^2\sigma\upsilon\nu 2x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}} 2x\sigma\upsilon\nu 2x dx\right)$$
$$=-\frac{1}{2}\left(0-\frac{1}{2}\int_0^{\frac{\pi}{4}} 2x\left(\eta\mu 2x\right)' dx\right)=-\frac{1}{2}\left(-\int_0^{\frac{\pi}{4}} x\left(\eta\mu 2x\right)' dx\right)=\frac{1}{2}\int_0^{\frac{\pi}{4}} x\left(\eta\mu 2x\right)' dx=$$
$$=\frac{1}{2}\left(\left[x\eta\mu 2x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\eta\mu 2x dx\right)=\frac{1}{2}\left(\frac{\pi}{4}\cdot 1-0+\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\sigma\upsilon\nu 2x\right)' dx\right)=$$
$$=\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\left[\sigma\upsilon\nu 2x\right]_0^{\frac{\pi}{4}}\right)=\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\left(0-1\right)\right)=\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{\pi}{8}-\frac{1}{4}$$

$$\Delta=\int_0^1\left(x^2+2x+1\right)e^{-2x}dx=-\frac{1}{2}\int_0^1\left(x^2+2x+1\right)\left(e^{-2x}\right)' dx=-\frac{1}{2}\left(\left[\left(x^2+2x+1\right)e^{-2x}\right]_0^1-\int_0^1\left(x^2+2x+1\right)' e^{-2x}dx\right)=$$
$$=-\frac{1}{2}\left(\frac{4}{e^2}-1-\int_0^1\left(2x+2\right)e^{-2x}dx\right)=-\frac{1}{2}\left(\frac{4}{e^2}-1+\frac{1}{2}\int_0^1 2\left(x+1\right)\left(e^{-2x}\right)' dx\right)=-\frac{1}{2}\left(\frac{4}{e^2}-1+\int_0^1\left(x+1\right)\left(e^{-2x}\right)' dx\right)=$$
$$=-\frac{1}{2}\left(\frac{4}{e^2}-1+\left[\left(x+1\right)e^{-2x}\right]_0^1-\int_0^1 e^{-2x}dx\right)=-\frac{1}{2}\left(\frac{4}{e^2}-1+\frac{2}{e^2}-1+\frac{1}{2}\int_0^1\left(e^{-2x}\right)' dx\right)=$$
$$=-\frac{1}{2}\left(\frac{6}{e^2}-2+\frac{1}{2}\left[e^{-2x}\right]_0^1\right)=-\frac{1}{2}\left(\frac{6}{e^2}-2+\frac{1}{2}\left(\frac{1}{e^2}-1\right)\right)=-\frac{1}{2}\left(\frac{6}{e^2}-2+\frac{1}{2e^2}-\frac{1}{2}\right)=$$
$$=-\frac{1}{2}\left(\frac{13}{2e^2}-\frac{5}{2}\right)=-\frac{13}{4e^2}+\frac{5}{4}$$

14.32

$$\mathrm{A}=\int_0^1 x\sqrt{1-x}dx$$

Θέτω $u=\sqrt{1-x}\Rightarrow du=\left(\sqrt{1-x}\right)' dx\Rightarrow du=-\frac{1}{2\sqrt{1-x}}dx\Rightarrow du=-\frac{1}{2u}dx\Rightarrow dx=-2udu$

$x=1\Rightarrow u_1=\sqrt{1-1}=0$

$x=0\Rightarrow u_1=\sqrt{1-0}=1$

$u=\sqrt{1-x}\Leftrightarrow u^2=1-x\Leftrightarrow x=1-u^2$

Επομένως $\mathrm{A}=\int_1^0\left(1-u^2\right)u\left(-2u\right)du=\int_1^0\left(-2u^2\left(1-u^2\right)\right)du=\int_1^0\left(-2u^2+2u^4\right)du=$

$$=-2\left[\frac{u^3}{3}\right]_1^0+2\left[\frac{u^5}{5}\right]_1^0=-2\left(0-\frac{1}{3}\right)+2\left(0-\frac{1}{5}\right)=\frac{2}{3}-\frac{2}{5}=\frac{4}{15}$$

$$\mathrm{B}=\int_{2}^{5}\frac{x}{\sqrt{x-1}}dx$$

Θέτω $u=\sqrt{x-1}\Rightarrow du=\left(\sqrt{x-1}\right)'dx\Rightarrow du=\frac{1}{2u}dx\Rightarrow dx=2udu$

$x=5\Rightarrow u_1=\sqrt{5-1}=2$

$x=2\Rightarrow u_1=\sqrt{2-1}=1$

$u=\sqrt{x-1}\Leftrightarrow u^2=x-1\Leftrightarrow x=1+u^2$

Επομένως $\mathrm{B}=\int_{1}^{2}\frac{1+u^2}{u}2udu=2\int_{1}^{2}\left(1+u^2\right)du=2\left[u+\frac{u^3}{3}\right]_1^2=2\left(2+\frac{8}{3}-1-\frac{1}{3}\right)=2\left(1+\frac{7}{3}\right)=\frac{20}{3}$

$$\Gamma=\int_{0}^{1}\frac{\sqrt{x}}{1+\sqrt{x}}dx$$

Θέτω $u=1+\sqrt{x}\Rightarrow du=\left(1+\sqrt{x}\right)'dx\Rightarrow du=\frac{1}{2\sqrt{x}}dx\Rightarrow du=\frac{1}{2\left(u-1\right)}dx\Rightarrow dx=2\left(u-1\right)du$

$x=1\Rightarrow u_1=1+\sqrt{x}=2$

$x=0\Rightarrow u_1=1+\sqrt{x}=1$

$u=1+\sqrt{x}\Leftrightarrow\sqrt{x}=u-1\Rightarrow x=\left(u-1\right)^2$

Επομένως $\Gamma=\int_{1}^{2}\frac{u-1}{u}2\left(u-1\right)du=2\int_{1}^{2}\frac{\left(u-1\right)^2}{u}du=2\int_{1}^{2}\frac{u^2-2u+1}{u}du=2\int_{1}^{2}\left(\frac{u^2}{u}-\frac{2u}{u}+\frac{1}{u}\right)du=$

$=2\int_{1}^{2}\left(u-2+\frac{1}{u}\right)du=2\left[\frac{u^2}{2}-2u+\ln\left|u\right|\right]_1^2=2\left(2-4+\ln 2-\frac{1}{2}+2-0\right)=2\left(\ln 2-\frac{1}{2}\right)=2\ln 2-1$

$$\Delta=\int_{0}^{\pi^2}\eta\mu\sqrt{x}dx$$

Θέτω $u=\sqrt{x}\Rightarrow du=\left(\sqrt{x}\right)'dx\Rightarrow du=\frac{1}{2\sqrt{x}}dx\Rightarrow du=\frac{1}{2u}dx\Rightarrow dx=2udu$

$x=\pi^2\Rightarrow u_1=\sqrt{\pi^2}=\pi$

$x=0\Rightarrow u_1=\sqrt{0}=0$

Επομένως $\Delta=2\int_{0}^{\pi}u\eta\mu udu=-2\int_{0}^{\pi}u\left(\sigma\upsilon\nu u\right)'du=-2\left(\left[u\sigma\upsilon\nu u\right]_0^{\pi}-\int_{0}^{\pi}\left(u\right)'\sigma\upsilon\nu udu\right)$

$=-2\left(-\pi-0-\int_{0}^{\pi}\sigma\upsilon\nu udu\right)=-2\left(-\pi-\left[\eta\mu u\right]_0^{\pi}\right)==-2\left(-\pi-0\right)=2\pi$

14.33

Αν οι συναρτήσεις f και g έχουν συνεχή δεύτερη παράγωγο και f(α) = f(β) = g(α) = g(β) = 0, να αποδείξετε ότι: $\int_{\alpha}^{\beta} f''(x)g(x)\,dx = \int_{\alpha}^{\beta} f(x)g''(x)\,dx$.

$$\int_{\alpha}^{\beta} f''(x)g(x)\,dx = \int_{\alpha}^{\beta} \left(f'(x)\right)' g(x)\,dx = \left[f'(x)g(x)\right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} f'(x)g'(x)\,dx =$$

$$= f'(\beta)g(\beta) - f'(\alpha)g(\alpha) - \int_{\alpha}^{\beta} f'(x)g'(x)\,dx = 0 - 0 - \int_{\alpha}^{\beta} f'(x)g'(x)\,dx = -\int_{\alpha}^{\beta} f'(x)g'(x)\,dx =$$

$$= -\left(\left[f(x)g'(x)\right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} f(x)g''(x)\,dx\right) = -\left(f(\beta)g'(\beta) - f(\alpha)g'(\alpha) - \int_{\alpha}^{\beta} f(x)g''(x)\,dx\right) =$$

$$= -\left(0 - 0 - \int_{\alpha}^{\beta} f(x)g''(x)\,dx\right) = \int_{\alpha}^{\beta} f(x)g''(x)\,dx$$

14.34

$$f(x) = \begin{cases} \ln x - 1 & , x \geq e \\ 1 - \ln x & , 0 < x < e \end{cases} \quad , \ \mathrm{A} = \int_{1}^{e^2} f(x)dx$$

f *συνεχής στο* $[1, \mathrm{e})$

f *συνεχής στο* $(e, \mathrm{e}^2]$

$\lim\limits_{x \to e^-} f(x) = \lim\limits_{x \to e^-} (1 - \ln x) = 0$ *οπότε* f *συνεχής στο* $[1, e^2]$

$\lim\limits_{x \to e^+} f(x) = \lim\limits_{x \to e^+} (\ln x - 1) = 0$

$f(\mathrm{e}) = \ln\mathrm{e} - 1 = 0$

$$\mathrm{A} = \int_{1}^{e^2} f(x)dx = \int_{1}^{e} f(x)dx + \int_{e}^{e^2} f(x)dx = \int_{1}^{e} (1 - \ln x)dx + \int_{e}^{e^2} (\ln x - 1)dx =$$

$$= \int_{1}^{e} 1dx - \int_{1}^{e} \ln x dx + \int_{e}^{e^2} \ln x dx - \int_{e}^{e^2} 1dx = e - 1 - \int_{1}^{e} \ln x dx + \int_{e}^{e^2} \ln x dx - \left(e^2 - e\right) =$$

$$= -e^2 + 2e - 1 - \int_{1}^{e} \ln x dx + \int_{e}^{e^2} \ln x dx = -e^2 + 2e - 1 - \int_{1}^{e} (x)' \ln x dx + \int_{e}^{e^2} (x)' \ln x dx =$$

$$= -e^2 + 2e - 1 - \left(\left[x \ln x\right]_{1}^{e} - \int_{1}^{e} x(\ln x)' dx\right) + \left[x \ln x\right]_{e}^{e^2} - \int_{e}^{e^2} x(\ln x)' dx =$$

$$= -e^2 + 2e - 1 - \left(e - 0 - \int_{1}^{e} 1dx\right) + 2e^2 - e - \int_{e}^{e^2} 1dx = -e^2 + 2e - 1 - e + \int_{1}^{e} 1dx + 2e^2 - e - \int_{e}^{e^2} 1dx =$$

$$= e^2 - 1 + \int_{1}^{e} 1dx - \int_{e}^{e^2} 1dx = e^2 - 1 + e - 1 - \left(e^2 - e\right) = 2e - 2$$

$$f(x) = 3\left|x^2 - 4\right| + 2x \quad , \ \mathrm{B} = \int_{0}^{4} f(x)dx$$

Είναι $f(x) = 3\left(4 - x^2\right) + 2x = -3x^2 + 2x + 12 \ , \ x \in (-2, 2)$

Είναι $f(x) = 3\left(x^2 - 4\right) + 2x = 3x^2 + 2x - 12 \ , \ x \in (-\infty, -2] \cup [2, +\infty)$

$$\mathrm{B} = \int_{0}^{4} f(x)dx = \int_{0}^{2} f(x)dx + \int_{2}^{4} f(x)dx = \int_{0}^{2} \left(-3x^2 + 2x + 12\right)dx + \int_{2}^{4} \left(3x^2 + 2x - 12\right)dx =$$

$$= \left[-x^3 + x^2 + 12x\right]_{0}^{2} + \left[x^3 + x^2 - 12x\right]_{2}^{4} = (-8 + 4 + 24) - 0 + (64 + 16 - 48) - (8 + 4 - 24) =$$

$$= 20 + 32 + 12 = 64$$

$f(x)=|\ln x| \quad , \quad \Gamma=\int_{\frac{1}{e}}^{e} f(x)dx$

Είναι $f(x)=\ln x \ , \ x\in[1,+\infty)$

Είναι $f(x)=-\ln x \ , \ x\in(0,1)$

f *συνεχής στο* $(0,+\infty)$

$$\Gamma=\int_{\frac{1}{e}}^{e} f(x)dx=\int_{\frac{1}{e}}^{1} f(x)dx+\int_{1}^{e} f(x)dx=\int_{\frac{1}{e}}^{1}(-\ln x)dx+\int_{1}^{e}\ln x\,dx=-\int_{\frac{1}{e}}^{1}\ln x\,dx+\int_{1}^{e}\ln x\,dx=$$

$$=-\int_{\frac{1}{e}}^{1}(x)'\ln x\,dx+\int_{1}^{e}(x)'\ln x\,dx=-\left([x\ln x]_{\frac{1}{e}}^{1}-\int_{\frac{1}{e}}^{1}x(\ln x)'\,dx\right)+[x\ln x]_{1}^{e}-\int_{1}^{e}x(\ln x)'\,dx=$$

$$=-\left(0+\frac{1}{e}-\int_{\frac{1}{e}}^{1}1dx\right)+e-0-\int_{1}^{e}1\,dx=-\frac{1}{e}+\int_{\frac{1}{e}}^{1}1dx+e-\int_{1}^{e}1\,dx=-\frac{1}{e}+1-\frac{1}{e}+e-e+1=-\frac{2}{e}+2$$

$f(x)=|e^x-x-1| \quad , \quad \Delta=\int_{-1}^{1} f(x)dx$

Είναι $e^x\geq x+1 \Leftrightarrow e^x-x-1\geq 0 \ , \ x\in R$

Είναι $f(x)=e^x-x-1 \ , \ x\in R$

f *συνεχής στο* R

$$\Delta=\int_{-1}^{1} f(x)dx=\int_{-1}^{1}(e^x-x-1)dx=\left[e^x-\frac{x^2}{2}-x\right]_{-1}^{1}=\left(e-\frac{1}{2}-1\right)-\left(\frac{1}{e}-\frac{1}{2}+1\right)=e-\frac{1}{e}-2$$

14.35

Αν η συνάρτηση f έχει συνεχή παράγωγο στο διάστημα [0,1] και:
$\int_0^1 x^2 f'(x)dx=2004-2\int_0^1 xf(x)dx$, να βρείτε την τιμή f(1).

$$\int_0^1 x^2 f'(x)dx=2004-2\int_0^1 xf(x)dx \Leftrightarrow$$

$$\int_0^1 x^2 f'(x)dx+2\int_0^1 xf(x)dx=2004 \Leftrightarrow$$

$$\int_0^1\left(x^2 f'(x)+2xf(x)\right)dx=2004 \Leftrightarrow$$

$$\int_0^1\left(x^2 f(x)\right)' dx=2004 \Leftrightarrow \left[x^2 f(x)\right]_0^1=2004 \Leftrightarrow f(1)=2004$$

14.36

Αν η συνάρτηση f έχει συνεχή παράγωγο στο διάστημα [0,2] και: $\int_0^2 xf'(x)dx=4\alpha-\int_0^2 f(x)dx$, να δείξετε ότι f(2)=2α.

$$\int_0^2 xf'(x)dx=4\alpha-\int_0^2 f(x)dx \Leftrightarrow \int_0^2 xf'(x)dx+\int_0^2 f(x)dx=4\alpha \Leftrightarrow$$

$$\int_0^2\left(xf'(x)+f(x)\right)dx=4\alpha \Leftrightarrow \int_0^2\left(xf(x)\right)'dx=4\alpha \Leftrightarrow \left[xf(x)\right]_0^2=4\alpha \Leftrightarrow$$

$$2f(2)=4\alpha \Leftrightarrow f(2)=2\alpha$$

14.37

$$A=\int_0^{\ln 2}\frac{e^x-1}{e^x+1}dx$$

Θέτω $u=e^x \Rightarrow du=e^x dx \Rightarrow du=udx \Rightarrow dx=\frac{1}{u}du$

$x=\ln 2 \Rightarrow u_2=e^{\ln 2}=2$

$x=0 \Rightarrow u_1=e^0=1$

Επομένως $A=\int_1^2\frac{u-1}{u+1}\cdot\frac{1}{u}du=\int_1^2\frac{u-1}{u(u+1)}du=\int_1^2\frac{(u+1)-2}{u(u+1)}du=\int_1^2\left(\frac{(u+1)}{u(u+1)}-\frac{2}{u(u+1)}\right)du=$

$=\int_1^2\frac{1}{u}du-\int_1^2\frac{2}{u(u+1)}du=$

$\frac{2}{u(u+1)}=\frac{A}{u}+\frac{B}{u+1}\Leftrightarrow 2=A(u+1)+Bu\Leftrightarrow 2=Au+A+Bu\Leftrightarrow (A+B)u+(A-2)=0\Leftrightarrow$

$\Leftrightarrow\begin{cases}A+B=0\\A-2=0\end{cases}\Leftrightarrow\begin{cases}B=-2\\A=2\end{cases}$

$=\int_1^2\frac{1}{u}du-\left(\int_1^2\frac{2}{u}du-\int_1^2\frac{2}{u+1}du\right)=\int_1^2\frac{1}{u}du-\int_1^2\frac{2}{u}du+\int_1^2\frac{2}{u+1}du=$

$=\left[\ln|u|\right]_1^2-2\left[\ln|u|\right]_1^2+2\left[\ln|u+1|\right]_1^2=$

$=(\ln 2-0)-2(\ln 2-0)+2(\ln 3-\ln 2)=-3\ln 2+2\ln 3$

$$\mathrm{B}=\int_3^8 \frac{1}{x\sqrt{x+1}}dx$$

$$\Theta\acute{\varepsilon}\tau\omega\ u=\sqrt{x+1}\Rightarrow du=\left(\sqrt{x+1}\right)'dx\Rightarrow du=\frac{1}{2u}dx\Rightarrow dx=2udu$$

$$x=8\Rightarrow u_1=\sqrt{8+1}=3$$

$$x=3\Rightarrow u_1=\sqrt{3+1}=2$$

$$u=\sqrt{x+1}\Leftrightarrow u^2=x+1\Leftrightarrow x=u^2-1$$

$$\text{Επομένως}\ \ \mathrm{B}=\int_2^3 \frac{1}{\left(u^2-1\right)u}2udu=2\int_2^3\frac{1}{u^2-1}du$$

$$\frac{1}{(u-1)(u+1)}=\frac{\mathrm{A}}{u-1}+\frac{\mathrm{B}}{u+1}\Leftrightarrow 1=\mathrm{A}(u+1)+\mathrm{B}(u-1)\Leftrightarrow 1=\mathrm{A}u+\mathrm{A}+\mathrm{B}u-\mathrm{B}$$

$$\Leftrightarrow(\mathrm{A}+\mathrm{B})u+(\mathrm{A}-\mathrm{B}-1)=0\Leftrightarrow\begin{cases}\mathrm{A}+\mathrm{B}=0\\ \mathrm{A}-\mathrm{B}=1\end{cases}\Leftrightarrow\begin{cases}\mathrm{B}=-\dfrac{1}{2}\\ \mathrm{A}=\dfrac{1}{2}\end{cases}$$

$$\mathrm{B}=2\left(\int_2^3\frac{\frac{1}{2}}{u-1}du-\int_2^3\frac{\frac{1}{2}}{u+1}du\right)=2\cdot\frac{1}{2}\left(\int_2^3\frac{1}{u-1}du-\int_2^3\frac{1}{u+1}du\right)=\int_2^3\frac{1}{u-1}du-\int_2^3\frac{1}{u+1}du=$$

$$=\left[\ln|u-1|\right]_2^3-\left[\ln|u+1|\right]_2^3=(\ln 2-0)-(\ln 4-\ln 3)=\ln 2-2\ln 2+\ln 3=\ln\frac{3}{2}$$

$$\Gamma=\int_5^{12}\frac{1+\sqrt{x+4}}{x}dx$$

$$\Theta\acute{\varepsilon}\tau\omega\ u=\sqrt{x+4}\Rightarrow du=\left(\sqrt{x+4}\right)'dx\Rightarrow du=\frac{1}{2u}dx\Rightarrow dx=2udu$$

$$x=12\Rightarrow u_2=\sqrt{12+4}=4$$

$$x=5\Rightarrow u_1=\sqrt{5+4}=3$$

$$u=\sqrt{x+4}\Leftrightarrow u^2=x+4\Leftrightarrow x=u^2-4$$

$$\text{Επομένως}\ \ \Gamma=\int_3^4\frac{1+u}{u^2-4}2udu=2\int_3^4\frac{u^2+u}{u^2-4}du=2\int_3^4\frac{\left(u^2-4\right)+\left(u+4\right)}{u^2-4}du=2\int_3^4\left(1+\frac{u+4}{u^2-4}\right)du$$

$$\frac{u+4}{(u-2)(u+2)}=\frac{A}{u-2}+\frac{B}{u+2}\Leftrightarrow u+4=A(u+2)+B(u-2)\Leftrightarrow u+4=Au+2A+Bu-2B$$

$$\Leftrightarrow (A+B-1)u+(2A-2B-4)=0\Leftrightarrow\begin{cases}A+B=1\\A-B=2\end{cases}\Leftrightarrow\begin{cases}B=-\dfrac{1}{2}\\ A=\dfrac{3}{2}\end{cases}$$

$$\Gamma=2\int_3^4 1du+2\left(\int_3^4\frac{\frac{3}{2}}{u-2}du-\int_3^4\frac{\frac{1}{2}}{u+2}du\right)=2+2\left(\frac{3}{2}\int_3^4\frac{1}{u-2}du-\frac{1}{2}\int_3^4\frac{1}{u+2}du\right)=$$

$$=2+3\int_3^4\frac{1}{u-2}du-\int_3^4\frac{1}{u+2}du=$$

$$=2+3\left[\ln|u-2|\right]_3^4-\left[\ln|u+2|\right]_3^4=2+3(\ln 2-0)-(\ln 6-\ln 5)=$$

$$=2+3\ln 2+\ln 5-\ln 6=2+\ln 8+\ln 5-\ln 6=2+\ln\frac{20}{3}$$

$$\Delta = \int_0^{\frac{\pi}{2}} \frac{\sigma\upsilon\nu x}{\eta\mu^2 x + 3\eta\mu x + 2} dx$$

Θέτω $u = \eta\mu x \Rightarrow du = (\eta\mu x)' dx \Rightarrow du = \sigma\upsilon\nu x dx$

$x = \frac{\pi}{2} \Rightarrow u_2 = \eta\mu\frac{\pi}{2} = 1$

$x = 0 \Rightarrow u_1 = \eta\mu 0 = 0$

Επομένως $\Delta = \int_0^1 \frac{1}{u^2 + 3u + 2} du = \int_0^1 \frac{1}{(u+1)(u+2)} du =$

$$\frac{1}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2} \Leftrightarrow 1 = A(u+2) + B(u+1) \Leftrightarrow 1 = Au + 2A + Bu + B$$

$$\Leftrightarrow (A+B)u + (2A+B-1) = 0 \Leftrightarrow \begin{cases} A+B=0 \\ 2A+B=1 \end{cases} \Leftrightarrow \begin{cases} B=-1 \\ A=1 \end{cases}$$

$$\Delta = \int_0^1 \frac{1}{u+1} du - \int_0^1 \frac{1}{u+2} du = \left[\ln|u+1|\right]_0^1 - \left[\ln|u+2|\right]_0^{11} =$$
$$= (\ln 2 - 0) - (\ln 3 - \ln 2) = 2\ln 2 - \ln 3$$

14.38

$$A = \int_0^{\frac{\pi}{2}} \left(\eta\mu^3 x + \eta\mu^4 x\right)\sigma\upsilon\nu x dx$$

Θέτω $u = \eta\mu x \Rightarrow du = (\eta\mu x)' dx \Rightarrow du = \sigma\upsilon\nu x dx$

$x = \frac{\pi}{2} \Rightarrow u_2 = \eta\mu\frac{\pi}{2} = 1$

$x = 0 \Rightarrow u_1 = \eta\mu 0 = 0$

Επομένως $A = \int_0^1 \left(u^3 + u^4\right) du = \frac{1}{4}\left[u^4\right]_0^1 + \frac{1}{5}\left[u^5\right]_0^1 = \frac{1}{4} + \frac{1}{5} = \frac{9}{20}$

$$B = \int_0^{\pi} \left(\sigma\upsilon\nu^4 x + \eta\mu^2 x\right)\eta\mu x dx = \int_0^{\pi} \left(\sigma\upsilon\nu^4 x + 1 - \sigma\upsilon\nu^2 x\right)\eta\mu x dx$$

Θέτω $u = \sigma\upsilon\nu x \Rightarrow du = (\sigma\upsilon\nu x)' dx \Rightarrow -du = \eta\mu x dx$

$x = \pi \Rightarrow u_2 = \sigma\upsilon\nu\pi = -1$

$x = 0 \Rightarrow u_1 = \sigma\upsilon\nu 0 = 1$

Επομένως $B = -\int_1^{-1} \left(u^4 + 1 - u^2\right) du = \int_{-1}^1 \left(u^4 + 1 - u^2\right) du =$

$$\frac{1}{5}\left[u^5\right]_{-1}^1 + [u]_{-1}^1 - \frac{1}{3}\left[u^3\right]_{-1}^1 = \frac{1}{5}(1+1) + (1+1) - \frac{1}{3}(1+1) = \frac{2}{5} + 2 - \frac{2}{3} = \frac{26}{15}$$

14.39

Αν η συνάρτηση f έχει συνεχή δεύτερη παράγωγο με f(π) = 3 και: $\int_0^{\pi}\left(f(x)+f''(x)\right)\eta\mu x dx = 7$, να δείξετε ότι f(0)=4.

$$\int_0^{\pi}\left(f(x)+f''(x)\right)\eta\mu x dx = 7 \Leftrightarrow$$

$$\int_0^{\pi} f(x)\eta\mu x dx + \int_0^{\pi} f''(x)\eta\mu x dx = 7 \Leftrightarrow$$

$$-\int_0^{\pi} f(x)(\sigma\upsilon\nu x)'\,dx + \int_0^{\pi}\left(f'(x)\right)'\eta\mu x dx = 7 \Leftrightarrow$$

$$-\left(\left[f(x)\sigma\upsilon\nu x\right]_0^{\pi} - \int_0^{\pi} f'(x)\sigma\upsilon\nu x dx\right) + \left[f'(x)\eta\mu x\right]_0^{\pi} - \int_0^{\pi} f'(x)(\eta\mu x)'\,dx = 7 \Leftrightarrow$$

$$-\left[f(x)\sigma\upsilon\nu x\right]_0^{\pi} + \int_0^{\pi} f'(x)\sigma\upsilon\nu x dx + \left[f'(x)\eta\mu x\right]_0^{\pi} - \int_0^{\pi} f'(x)\sigma\upsilon\nu x dx = 7 \Leftrightarrow$$

$$-\left[f(x)\sigma\upsilon\nu x\right]_0^{\pi} + \left[f'(x)\eta\mu x\right]_0^{\pi} = 7 \Leftrightarrow$$

$$-\left(f(\pi)\sigma\upsilon\nu\pi - f(0)\sigma\upsilon\nu 0\right) + f'(\pi)\eta\mu\pi - f'(0)\eta\mu 0 = 7 \Leftrightarrow$$

$$-\left(-3-f(0)\right) = 7 \Leftrightarrow 3+f(0) = 7 \Leftrightarrow f(0) = 4$$

14.40

$$\mathrm{A} = \int_0^1 \frac{2x^3-3x^2-11x+1}{x^2-x-6}dx = \int_0^1 \frac{(2x-1)(x^2-x-6)-5}{x^2-x-6}dx =$$

$$= \int_0^1\left(\frac{(2x-1)(x^2-x-6)}{x^2-x-6} - \frac{5}{x^2-x-6}\right)dx = \int_0^1\left((2x-1) - \frac{5}{x^2-x-6}\right)dx =$$

$$= \int_0^1\left((2x-1) - \frac{5}{(x+2)(x-3)}\right)dx =$$

$$\frac{5}{(x+2)(x-3)} = \frac{\mathrm{A}}{x+2} + \frac{\mathrm{B}}{x-3} \Leftrightarrow 5 = \mathrm{A}(x-3) + \mathrm{B}(x+2) \Leftrightarrow 5 = \mathrm{A}x - 3\mathrm{A} + \mathrm{B}x + 2\mathrm{B}$$

$$\Leftrightarrow (\mathrm{A}+\mathrm{B})x + (-3\mathrm{A}+2\mathrm{B}-5) = 0 \Leftrightarrow \begin{cases} \mathrm{A}+\mathrm{B}=0 \\ -3\mathrm{A}+2\mathrm{B}=5 \end{cases} \Leftrightarrow \begin{cases} \mathrm{B}=1 \\ \mathrm{A}=-1 \end{cases}$$

$$\mathrm{A} = \int_0^1 (2x-1)dx - \left(-\int_0^1 \frac{1}{x+2}dx + \int_0^1 \frac{1}{x-3}dx\right) =$$

$$= \left[x^2-x\right]_0^1 + \left[\ln|x+2|\right]_0^1 - \left[\ln|x-3|\right]_0^1 = (0-0) + (\ln 3 - \ln 2) - (\ln 2 - \ln 3) = -2\ln 2 + 2\ln 3 = 2\ln\frac{3}{2}$$

14.41

Δίνεται η συνάρτηση f για την οποία ισχύει $xf(x)-f(-x)=x+1 \quad ,x\in R$

Να υπολογίσετε το ολοκλήρωμα $\mathrm{I}=\int_0^1 f(x)dx$

$$xf(x)-f(-x)=x+1 \quad ,x\in R \quad (1)$$

$$\text{για} \quad x \quad \text{το} \quad -x:\ (1)\Rightarrow -xf(-x)-f(x)=-x+1 \qquad (2)$$

$$\text{πολλαπλασιάζοντας επί } -x:(1)\Rightarrow -x^2f(x)+xf(-x)=-x^2-x \quad (3)$$

$$(2)+(3)\Rightarrow -x^2f(x)-f(x)=-x^2-x-x+1$$

$$\Rightarrow\left(-x^2-1\right)f(x)=-x^2-2x+1$$

$$\Rightarrow f(x)=\frac{-x^2-2x+1}{-x^2-1}$$

$$\Rightarrow f(x)=\frac{x^2+2x-1}{x^2+1}$$

$$\mathrm{I}=\int_0^1 f(x)dx=\int_0^1\frac{x^2+2x-1}{x^2+1}dx=\int_0^1\frac{\left(x^2+1\right)+2x-2}{x^2+1}dx=$$

$$=\int_0^1\frac{x^2+1}{x^2+1}dx+2\int_0^1\frac{x}{x^2+1}dx-2\int_0^1\frac{1}{x^2+1}dx$$

$$=\int_0^1 1dx+2\int_0^1\frac{x}{x^2+1}dx-2\int_0^1\frac{1}{x^2+1}dx$$

$$*\int_0^1 1dx=1\cdot(1-0)=1$$

$$*\int_0^1\frac{x}{x^2+1}dx=\frac{1}{2}\int_0^1\frac{2x}{x^2+1}dx=\frac{1}{2}\left[\ln\left(x^2+1\right)\right]_0^1=\frac{1}{2}(\ln 2-0)=\frac{1}{2}\ln 2$$

$$*\int_0^1\frac{1}{x^2+1}dx=$$

$$\text{Θέτω } x=\varepsilon\varphi u\Rightarrow dx=\frac{1}{\sigma\upsilon\nu^2 u}du$$

$$x=1\Rightarrow \varepsilon\varphi u_2=0\Rightarrow u_2=\frac{\pi}{4}$$

$$x=0\Rightarrow \varepsilon\varphi u_1=0\Rightarrow u_1=0$$

$$\int_0^1\frac{1}{x^2+1}dx=\int_0^{\frac{\pi}{4}}\frac{1}{\varepsilon\varphi^2u+1}\cdot\frac{1}{\sigma\upsilon\nu^2u}du=\int_0^{\frac{\pi}{4}}\frac{1}{\frac{1}{\sigma\upsilon\nu^2u}}\cdot\frac{1}{\sigma\upsilon\nu^2u}du=\int_0^{\frac{\pi}{4}}1du=\frac{\pi}{4}$$

$$\text{Οπότε}:\ \mathrm{I}=1+2\cdot\frac{1}{2}\ln 2-2\cdot\frac{\pi}{4}=1+\ln 2-\frac{\pi}{2}$$

14.42

Έστω f:R→R παραγωγίσιμη συνάρτηση με $f'(x)=2e^{x^2}$, $x\in R$ και f(1) = e.

Να υπολογίσετε το ολοκλήρωμα $I=\int_0^1 f(x)dx$

$$I=\int_0^1 f(x)dx=\int_0^1 (x)' f(x)dx=\left[xf(x)\right]_0^1-\int_0^1 xf'(x)dx=f(1)-\int_0^1 x\cdot 2e^{x^2}dx=$$

$$=e-\int_0^1 (x^2)'\cdot e^{x^2}dx=e-\int_0^1 \left(e^{x^2}\right)'dx=e-\left[e^{x^2}\right]_0^1=e-(e-1)=1$$

14.43

α. Να αποδείξετε ότι $\int_0^1 x^3e^{x^2}dx=\frac{1}{2}$

$$\int_0^1 x^3e^{x^2}dx=\int_0^1 \left(x^2\cdot x\cdot e^{x^2}\right)dx=\frac{1}{2}\int_0^1 x^2\left(2x\cdot e^{x^2}\right)dx=\frac{1}{2}\int_0^1 x^2\left(e^{x^2}\right)'dx=$$

$$=\frac{1}{2}\left(\left[x^2e^{x^2}\right]_0^1-\int_0^1 (x^2)'e^{x^2}dx\right)=\frac{1}{2}\left(e-0-\int_0^1 2xe^{x^2}dx\right)=\frac{1}{2}\left(e-\int_0^1 \left(e^{x^2}\right)'dx\right)=$$

$$=\frac{1}{2}\left(e-\left[e^{x^2}\right]_0^1\right)=\frac{1}{2}\left(e-(e-1)\right)=\frac{1}{2}$$

β. Να αποδείξετε ότι $\int_0^1 x^5e^{x^3}dx=\frac{1}{3}$

$$\int_0^1 x^5e^{x^3}dx=\int_0^1 x^3x^2e^{x^3}dx=\frac{1}{3}\int_0^1 x^3\cdot 3x^2e^{x^3}dx=\frac{1}{3}\int_0^1 x^3\cdot\left(e^{x^3}\right)'dx=$$

$$=\frac{1}{3}\left(\left[x^3e^{x^3}\right]_0^1-\int_0^1 (x^3)'e^{x^3}dx\right)=\frac{1}{3}\left(e-0-\int_0^1 3x^2e^{x^3}dx\right)=\frac{1}{3}\left(e-\int_0^1 \left(e^{x^3}\right)'dx\right)=$$

$$=\frac{1}{3}\left(e-\left[e^{x^3}\right]_0^1\right)=\frac{1}{3}\left(e-(e-1)\right)=\frac{1}{3}$$

14.44

$A=\int_0^1 \frac{1}{x^2+1}dx=\frac{\pi}{4}$ - όπως στην άσκηση **14.41**

$\mathrm{B} = \int_0^{\ln 2} \sqrt{e^x - 1}dx$

Θέτω $u = \sqrt{e^x - 1} \Rightarrow du = \frac{1}{2\sqrt{e^x - 1}} e^x dx \Rightarrow du = \frac{1}{2u}\left(u^2 + 1\right)dx \Rightarrow dx = \frac{2u}{u^2 + 1}du$

$x = \ln 2 \Rightarrow u_2 = \sqrt{e^{\ln 2} - 1} = 1$

$x = 0 \Rightarrow u_1 = \sqrt{e^0 - 1} = 0$

$u = \sqrt{e^x - 1} \Rightarrow u^2 = e^x - 1 \Rightarrow e^x = u^2 + 1$

Επομένως $\mathrm{B} = \int_0^1 u \cdot \frac{2u}{u^2 + 1}du = 2\int_0^1 \frac{u^2}{u^2 + 1}du = 2\int_0^1 \frac{\left(u^2 + 1\right) - 1}{u^2 + 1}du =$

$= 2\left(\int_0^1 1du - \int_0^1 \frac{1}{u^2 + 1}du\right) \overset{*}{=} 2\left(1 - \frac{\pi}{4}\right) = 2 - \frac{\pi}{2}$

14.45

$\mathrm{A} = \int_1^e \frac{\ln x}{x\left(1 + \sqrt{\ln x}\right)}dx$

Θέτω $u = \sqrt{\ln x} \Rightarrow du = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x}dx \Rightarrow du = \frac{1}{2u} \cdot \frac{1}{x}dx \Rightarrow \frac{1}{x}dx = 2udu$

$x = e \Rightarrow u_2 = \sqrt{\ln e} = 1$

$x = 1 \Rightarrow u_1 = \sqrt{\ln 1} = 0$

$u = \sqrt{\ln x} \Rightarrow u^2 = \ln x$

Επομένως $\mathrm{A} = \int_0^1 \frac{u^2}{1 + u}2udu = 2\int_0^1 \frac{u^3}{u + 1}du = 2\int_0^1 \frac{\left(u^3 + 1\right) - 1}{u + 1}du =$

$= 2\int_0^1 \frac{\left(u + 1\right)\left(u^2 - u + 1\right) - 1}{u + 1}du = 2\left(\int_0^1 \left(u^2 - u + 1\right)du - \int_0^1 \frac{1}{u + 1}du\right) =$

$= 2\left(\left[\frac{u^3}{3} - \frac{u^2}{2} + u\right]_0^1 - \left[\ln|u + 1|\right]_0^1\right) = 2\left(\left(\frac{1}{3} - \frac{1}{2} + 1 - 0\right) - \left(\ln 2 - 0\right)\right) =$

$= 2\left(\frac{1}{3} - \frac{1}{2} + 1 - \ln 2\right) = \frac{5}{3} - 2\ln 2$

$$\mathrm{B}=\int_{\ln 3}^{3\ln 2}\sqrt{e^x+1}\,dx$$

$$\Theta\acute{\varepsilon}\tau\omega\ u=e^x+1\Rightarrow du=e^x dx\Rightarrow du=(u-1)dx\Rightarrow dx=\frac{1}{u-1}du$$

$$x=3\ln 2=\ln 8\Rightarrow u_2=e^{\ln 8}+1=9$$

$$x=\ln 3\Rightarrow u_1=e^{\ln 3}+1=4$$

$$u=e^x+1\Rightarrow e^x=u-1$$

$$\mathrm{E}\pi o\mu\acute{\varepsilon}\nu\omega\varsigma\ \ \mathrm{B}=\int_4^9\sqrt{u}\cdot\frac{1}{u-1}du=\int_4^9\frac{\sqrt{u}}{u-1}du=\int_4^9\frac{\left(\sqrt{u}-1\right)+1}{u-1}du=$$

$$=\int_4^9\frac{\sqrt{u}-1}{u-1}du+\int_4^9\frac{1}{u-1}du=\int_4^9\frac{\left(\sqrt{u}-1\right)\left(\sqrt{u}+1\right)}{(u-1)\left(\sqrt{u}+1\right)}du+\left[\ln|u-1|\right]_4^9=$$

$$=\int_4^9\frac{1}{\sqrt{u}+1}du+\left(\ln 8-\ln 3\right)\overset{*}{=}\left(2-2\ln 4+2\ln 3\right)+\ln 8-\ln 3$$

$$=2-4\ln 2+2\ln 3+3\ln 2-\ln 3=2+\ln\frac{3}{2}.$$

$$*\int_4^9\frac{1}{\sqrt{u}+1}du=$$

$$\Theta\acute{\varepsilon}\tau\omega\ \ y=\sqrt{u}+1\Rightarrow dy=\frac{1}{2\sqrt{u}}du\Rightarrow dy=\frac{1}{2(y-1)}du\Rightarrow du=2(y-1)dy$$

$$u=9\Rightarrow y_2=\sqrt{9}+1=4$$

$$u=4\Rightarrow y_1=\sqrt{4}+1=3$$

$$y=\sqrt{u}+1\Rightarrow\sqrt{u}=y-1$$

$$\acute{A}\rho\alpha\ \int_4^9\frac{1}{\sqrt{u}+1}du=\int_3^4\frac{1}{y}2(y-1)dy=2\int_3^4\frac{y-1}{y}dy=2\left(\int_3^4 1dy-\int_3^4\frac{1}{y}dy\right)=$$

$$=2\left(1-\left(\ln 4-\ln 3\right)\right)=2-2\ln 4+2\ln 3$$

14.46

Να αποδείξετε τους παρακάτω αναγωγικούς τύπους:

α. Αν $I_\nu = \int_0^{\frac{\pi}{2}} \eta\mu^\nu x dx, \quad \tau\acute{o}\tau\varepsilon \ \nu \cdot I_\nu = (\nu - 1)\cdot I_{\nu-2} \quad , \nu \geq 2$

$$I_\nu = \int_0^{\frac{\pi}{2}} \eta\mu^\nu x dx = \int_0^{\frac{\pi}{2}} \eta\mu^{\nu-1} x \cdot \eta\mu x dx = -\int_0^{\frac{\pi}{2}} \eta\mu^{\nu-1} x \cdot (\sigma\upsilon\nu x)' dx$$

$$I_\nu = -\left(\left[\eta\mu^{\nu-1} x \cdot \sigma\upsilon\nu x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(\eta\mu^{\nu-1} x \right)' \cdot \sigma\upsilon\nu x dx \right)$$

$$I_\nu = -\left(0 - 0 - (\nu - 1) \int_0^{\frac{\pi}{2}} \eta\mu^{\nu-2} x \cdot (\eta\mu x)' \cdot \sigma\upsilon\nu x dx \right)$$

$$I_\nu = (\nu - 1) \int_0^{\frac{\pi}{2}} \eta\mu^{\nu-2} x \cdot \sigma\upsilon\nu^2 x dx$$

$$I_\nu = (\nu - 1) \int_0^{\frac{\pi}{2}} \eta\mu^{\nu-2} x \cdot \left(1 - \eta\mu^2 x\right) dx$$

$$I_\nu = (\nu - 1) \left(\int_0^{\frac{\pi}{2}} \eta\mu^{\nu-2} x dx - \int_0^{\frac{\pi}{2}} \eta\mu^\nu x dx \right)$$

$$I_\nu = (\nu - 1)\left(I_{\nu-2} - I_\nu\right)$$

$$I_\nu = (\nu - 1) I_{\nu-2} - (\nu - 1) I_\nu$$

$$I_\nu + (\nu - 1) I_\nu = (\nu - 1) I_{\nu-2}$$

$$\nu \cdot I_\nu = (\nu - 1) I_{\nu-2}$$

β. Αν $I_\nu = \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^\nu x dx, \quad \tau\acute{o}\tau\varepsilon \ \nu \cdot I_\nu = (\nu - 1)\cdot I_{\nu-2} \quad , \nu \geq 2$

$$I_\nu = \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^\nu x dx = \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^{\nu-1} x \cdot \sigma\upsilon\nu x dx = \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^{\nu-1} x \cdot (\eta\mu x)' dx$$

$$I_\nu = \left[\sigma\upsilon\nu^{\nu-1} x \cdot \eta\mu x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(\sigma\upsilon\nu^{\nu-1} x \right)' \cdot \eta\mu x dx$$

$$I_\nu = 0 - 0 - (\nu - 1) \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^{\nu-2} x \cdot (\sigma\upsilon\nu x)' \cdot \eta\mu x dx$$

$$I_\nu = (\nu - 1) \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^{\nu-2} x \cdot \eta\mu^2 x dx$$

$$I_\nu = (\nu - 1) \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^{\nu-2} x \cdot \left(1 - \sigma\upsilon\nu^2 x\right) dx$$

$$I_\nu = (\nu - 1) \left(\int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^{\nu-2} x dx - \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^\nu x dx \right)$$

$$I_\nu = (\nu - 1)\left(I_{\nu-2} - I_\nu\right)$$

$$I_\nu = (\nu - 1) I_{\nu-2} - (\nu - 1) I_\nu$$

$$I_\nu + (\nu - 1) I_\nu = (\nu - 1) I_{\nu-2}$$

$$\nu \cdot I_\nu = (\nu - 1) I_{\nu-2}$$

14.47

Αν είναι $I_\nu = \int_1^e \ln^\nu x dx$, $K_\nu = \int_1^e x \ln^\nu x dx$, $\nu \in N^*$, να αποδείξετε ότι:

α. $\boxed{I_\nu = e - \nu \cdot I_{\nu-1} \quad , \nu \geq 2}$

$$I_\nu = \int_1^e \ln^\nu x dx = \int_1^e (x)' \ln^\nu x dx = \left[x \ln^\nu x\right]_1^e - \int_1^e x\left(\ln^\nu x\right)' dx$$

$$I_\nu = e - 0 - \nu \int_1^e x \ln^{\nu-1} x (\ln x)' dx$$

$$I_\nu = e - \nu \int_1^e \ln^{\nu-1} dx$$

$$I_\nu = e - \nu \cdot I_{\nu-1}$$

β. $\boxed{2K_\nu + \nu \cdot K_{\nu-1} = e^2 \quad , \nu \geq 2}$

$$K_\nu = \int_1^e x \ln^\nu x dx = \int_1^e \left(\frac{x^2}{2}\right)' \ln^\nu x dx = \left[\frac{x^2}{2} \ln^\nu x\right]_1^e - \int_1^e \frac{x^2}{2}\left(\ln^\nu x\right)' dx$$

$$K_\nu = \frac{e^2}{2} - 0 - \nu \int_1^e \frac{x^2}{2} \ln^{\nu-1} x (\ln x)' dx$$

$$K_\nu = \frac{e^2}{2} - \frac{\nu}{2} \int_1^e x \ln^{\nu-1} x dx$$

$$K_\nu = \frac{e^2}{2} - \frac{\nu}{2} \cdot K_{\nu-1}$$

$$2K_\nu = e^2 - \nu \cdot K_{\nu-1}$$

$$2K_\nu + \nu \cdot K_{\nu-1} = e^2$$

14.48

Αν είναι $I_\nu = \int_0^{\frac{\pi}{4}} \varepsilon\varphi^\nu x dx$, $A_\nu = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sigma\varphi^\nu x dx$, να αποδείξετε ότι για κάθε φυσικό ν ≥ 3 ισχύει:

α. $\boxed{I_\nu = \frac{1}{\nu - 1} - I_{\nu-2}}$

$$I_\nu = \int_0^{\frac{\pi}{4}} \varepsilon\varphi^\nu x dx = \int_0^{\frac{\pi}{4}} \varepsilon\varphi^2 x \cdot \varepsilon\varphi^{\nu-2} x dx = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\sigma\upsilon\nu^2 x} - 1\right) \cdot \varepsilon\varphi^{\nu-2} x dx$$

$$I_\nu = \int_0^{\frac{\pi}{4}} \frac{1}{\sigma\upsilon\nu^2 x} \cdot \varepsilon\varphi^{\nu-2} x dx - \int_0^{\frac{\pi}{4}} \varepsilon\varphi^{\nu-2} x dx$$

$$I_\nu = \int_0^{\frac{\pi}{4}} (\varepsilon\varphi x)' \cdot \varepsilon\varphi^{\nu-2} x dx - I_{\nu-2}$$

$$I_\nu = \frac{1}{\nu - 1} \int_0^{\frac{\pi}{4}} \left(\varepsilon\varphi^{\nu-1} x\right)' dx - I_{\nu-2}$$

$$I_\nu = \frac{1}{\nu - 1} \cdot \left[\varepsilon\varphi^{\nu-1} x\right]_0^{\frac{\pi}{4}} - I_{\nu-2}$$

$$I_\nu = \frac{1}{\nu - 1} \cdot (1 - 0) - I_{\nu-2}$$

$$I_\nu = \frac{1}{\nu - 1} - I_{\nu-2}$$

β. $\boxed{A_\nu = \frac{1}{\nu - 1} - A_{\nu-2}}$

$$A_\nu = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sigma\varphi^\nu x dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sigma\varphi^2 x \cdot \sigma\varphi^{\nu-2} x dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{\eta\mu^2 x} - 1\right) \cdot \sigma\varphi^{\nu-2} x dx$$

$$A_\nu = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\eta\mu^2 x} \cdot \sigma\varphi^{\nu-2} x dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sigma\varphi^{\nu-2} x dx$$

$$A_\nu = -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sigma\varphi x)' \cdot \sigma\varphi^{\nu-2} x dx - A_{\nu-2}$$

$$A_\nu = -\frac{1}{\nu - 1} \int_0^{\frac{\pi}{4}} \left(\sigma\varphi^{\nu-1} x\right)' dx - A_{\nu-2}$$

$$A_\nu = -\frac{1}{\nu - 1} \cdot \left[\sigma\varphi^{\nu-1} x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - A_{\nu-2}$$

$$A_\nu = -\frac{1}{\nu - 1} \cdot (0 - 1) - A_{\nu-2}$$

$$A_\nu = \frac{1}{\nu - 1} - A_{\nu-2}$$

14.49

Αν είναι $\mathrm{I}_\nu = \int_{-1}^{1}\left(1-x^2\right)^\nu dx$, να αποδείξετε ότι: $\mathrm{I}_\nu = \frac{2\nu}{2\nu+1}\cdot \mathrm{I}_{\nu-1} \quad ,\nu \geq 2$

$$\mathrm{I}_\nu = \int_{-1}^{1}\left(1-x^2\right)^\nu dx = \int_{-1}^{1}(x)'\left(1-x^2\right)^\nu dx = \left[x\left(1-x^2\right)^\nu\right]_{-1}^{1} - \int_{-1}^{1} x\left(\left(1-x^2\right)^\nu\right)' dx$$

$$\mathrm{I}_\nu = 0-0-\nu\cdot\int_{-1}^{1} x\left(1-x^2\right)^{\nu-1}\cdot(-2x)\,dx$$

$$\mathrm{I}_\nu = 2\nu\cdot\int_{-1}^{1} x^2\left(1-x^2\right)^{\nu-1} dx$$

$$\mathrm{I}_\nu = 2\nu\cdot\int_{-1}^{1} x^2\left(1-x^2\right)^{\nu-1} dx$$

$$\mathrm{I}_\nu = 2\nu\cdot\int_{-1}^{1}\left(1-\left(1-x^2\right)\right)\left(1-x^2\right)^{\nu-1} dx$$

$$\mathrm{I}_\nu = 2\nu\cdot\left(\int_{-1}^{1}\left(1-x^2\right)^{\nu-1} dx - \int_{-1}^{1}\left(1-x^2\right)\left(1-x^2\right)^{\nu-1} dx\right)$$

$$\mathrm{I}_\nu = 2\nu\cdot\left(\int_{-1}^{1}\left(1-x^2\right)^{\nu-1} dx - \int_{-1}^{1}\left(1-x^2\right)^{\nu} dx\right)$$

$$\mathrm{I}_\nu = 2\nu\cdot\left(\mathrm{I}_{\nu-1} - \mathrm{I}_\nu\right)$$

$$\mathrm{I}_\nu = 2\nu\cdot \mathrm{I}_{\nu-1} - 2\nu\cdot \mathrm{I}_\nu$$

$$\mathrm{I}_\nu + 2\nu\cdot \mathrm{I}_\nu = 2\nu\cdot \mathrm{I}_{\nu-1}$$

$$\left(1+2\nu\right)\cdot \mathrm{I}_\nu = 2\nu\cdot \mathrm{I}_{\nu-1}$$

$$\mathrm{I}_\nu = \frac{2\nu}{2\nu+1}\mathrm{I}_{\nu-1}$$

14.50

Αν είναι $I_\nu = \int_1^e \frac{\ln^\nu x}{x^2}dx$, $\nu \in N^*$, να αποδείξετε ότι:

α. $I_\nu = -\frac{1}{e} + \nu \cdot I_{\nu-1} \quad , \nu \geq 2$

$$I_\nu = \int_1^e \frac{\ln^\nu x}{x^2}dx = \int_1^e \left(-\frac{1}{x}\right)' \ln^\nu x dx = \left[-\frac{1}{x}\cdot \ln^\nu x\right]_1^e - \int_1^e \left(-\frac{1}{x}\right)\left(\ln^\nu x\right)' dx$$

$$I_\nu = -\frac{1}{e} - 0 + \nu \int_1^e \frac{1}{x}\ln^{\nu-1} x \left(\ln x\right)' dx$$

$$I_\nu = -\frac{1}{e} + \nu \int_1^e \frac{1}{x^2}\ln^{\nu-1} x dx$$

$$I_\nu = -\frac{1}{e} + \nu \cdot I_{\nu-1}$$

β. $I_3 = 6 - \frac{16}{e}$

$$I_\nu = -\frac{1}{e} + \nu \cdot I_{\nu-1} \quad , \nu \geq 2$$

$$I_2 = -\frac{1}{e} + 2\cdot I_1$$

$$I_3 = -\frac{1}{e} + 3\cdot I_2 \Rightarrow I_3 = -\frac{1}{e} + 3\cdot\left(-\frac{1}{e} + 2\cdot I_1\right) = -\frac{4}{e} + 6\cdot I_1 \overset{*}{=} -\frac{4}{e} + 6\cdot\left(-\frac{2}{e} + 1\right) =$$

$$= -\frac{4}{e} - \frac{12}{e} + 6 = -\frac{16}{e} + 6$$

$$^*I_1 = \int_1^e \frac{\ln x}{x^2}dx = \int_1^e \left(-\frac{1}{x}\right)' \ln x dx = \left[-\frac{1}{x}\cdot \ln x\right]_1^e - \int_1^e \left(-\frac{1}{x}\right)\left(\ln x\right)' dx$$

$$= -\frac{1}{e} - 0 + \int_1^e \frac{1}{x^2}dx = -\frac{1}{e} + \left[-\frac{1}{x}\right]_1^e = -\frac{1}{e} + \left(-\frac{1}{e} + 1\right) = -\frac{2}{e} + 1$$

14.51

α. Αν είναι $\mathrm{I}_\nu = \int_1^e \mathrm{x}^2 \cdot \ln^\nu x dx$, να αποδείξετε ότι: $3\mathrm{I}_\nu = e^3 - \nu \cdot \mathrm{I}_{\nu-1}$, $\nu \geq 2$

$$\mathrm{I}_\nu = \int_1^e \mathrm{x}^2 \cdot \ln^\nu x dx = \int_1^e \left(\frac{x^3}{3}\right)' \cdot \ln^\nu x dx = \left[\frac{x^3}{3} \cdot \ln^\nu x\right]_1^e - \int_1^e \frac{x^3}{3} \cdot \left(\ln^\nu x\right)' dx$$

$$\mathrm{I}_\nu = \frac{e^3}{3} - 0 - \nu \int_1^e \frac{x^3}{3} \cdot \ln^{\nu-1} x \left(\ln x\right)' dx$$

$$\mathrm{I}_\nu = \frac{e^3}{3} - \frac{\nu}{3} \int_1^e x^2 \cdot \ln^{\nu-1} x dx$$

$$\mathrm{I}_\nu = \frac{e^3}{3} - \frac{\nu}{3} \cdot \mathrm{I}_{\nu-1}$$

$$3\mathrm{I}_\nu = e^3 - \nu \cdot \mathrm{I}_{\nu-1}$$

β. Αν είναι $\mathrm{I}_\nu = \int_0^1 \mathrm{x}^\nu \sqrt{1-x} dx$, να αποδείξετε ότι: $3\mathrm{I}_\nu = \frac{2\nu}{2\nu+3} \cdot \mathrm{I}_{\nu-1}$, $\nu \geq 2$

$$\sqrt{1-x} = (1-x)^{\frac{1}{2}} = \left(-\frac{(1-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)' = \left(-\frac{2}{3} \cdot (1-x)^{\frac{3}{2}}\right)' = \left(-\frac{2}{3} \cdot (1-x)(1-x)^{\frac{1}{2}}\right)' =$$

$$= \left(-\frac{2}{3} \cdot (1-x)\sqrt{1-x}\right)'$$

$$\mathrm{I}_\nu = \int_0^1 \mathrm{x}^\nu \sqrt{1-x} dx = \int_0^1 \mathrm{x}^\nu \left(-\frac{2}{3} \cdot (1-x)\sqrt{1-x}\right)' dx = -\frac{2}{3} \int_0^1 \mathrm{x}^\nu \left((1-x)\sqrt{1-x}\right)' dx$$

$$\mathrm{I}_\nu = -\frac{2}{3}\left(\left[\mathrm{x}^\nu (1-x)\sqrt{1-x}\right]_0^1 - \int_0^1 \left(\mathrm{x}^\nu\right)' (1-x)\sqrt{1-x} dx\right)$$

$$\mathrm{I}_\nu = -\frac{2}{3}\left(0 - 0 - \nu \int_0^1 \mathrm{x}^{\nu-1} (1-x)\sqrt{1-x} dx\right)$$

$$\mathrm{I}_\nu = \frac{2\nu}{3} \int_0^1 \mathrm{x}^{\nu-1} (1-x)\sqrt{1-x} dx$$

$$\mathrm{I}_\nu = \frac{2\nu}{3}\left(\int_0^1 \mathrm{x}^{\nu-1} \sqrt{1-x} dx - \int_0^1 \mathrm{x}^\nu \sqrt{1-x} dx\right)$$

$$\mathrm{I}_\nu = \frac{2\nu}{3}\left(\mathrm{I}_{\nu-1} - \mathrm{I}_\nu\right)$$

$$3\mathrm{I}_\nu = 2\nu \cdot \mathrm{I}_{\nu-1} - 2\nu \cdot \mathrm{I}_\nu$$

$$\mathrm{I}_\nu = \frac{2\nu}{2\nu+3} \cdot \mathrm{I}_{\nu-1}$$

14.52

Δίνεται η συνεχής συνάρτηση f : R→R για την οποία ισχύει: $f(x)+f(1-x)=4012\ , x\in R$

α. Να αποδείξετε ότι $\int_0^1 f(1-x)dx=\int_0^1 f(x)dx$

$$\int_0^1 f(1-x)dx=-\int_1^0 f(u)du=\int_0^1 f(u)du=\int_0^1 f(x)dx$$

Θέτω $u=1-x\Rightarrow du=-dx$

$x=1\Rightarrow u_2=1-1=0$

$x=0\Rightarrow u_1=1-0=1$

β. Να υπολογίσετε το $A=\int_0^1 f(x)dx$

$$f(x)+f(1-x)=4012\Rightarrow$$

$$\int_0^1\left(f(x)+f(1-x)\right)dx=\int_0^1 4012dx\Rightarrow$$

$$\int_0^1 f(x)dx+\int_0^1 f(1-x)dx=\left[4012x\right]_0^1\Rightarrow$$

$$\int_0^1 f(x)dx+\int_0^1 f(x)dx=4012\Rightarrow$$

$$2\int_0^1 f(x)dx=4012\Rightarrow$$

$$\int_0^1 f(x)dx=2006$$

14.53

Αν οι συναρτήσεις f , g είναι συνεχείς στο R , να αποδείξετε ότι:

α. $\boxed{\int_0^\alpha f(x)g(\alpha-x)dx=\int_0^\alpha f(\alpha-x)g(x)dx}$

$$\int_0^\alpha f(x)g(\alpha-x)dx=-\int_\alpha^0 f(\alpha-u)g(u)du=\int_0^\alpha f(\alpha-u)g(u)du$$

$$=\int_0^\alpha f(\alpha-x)g(x)dx$$

Θέτω $u=\alpha-x\Rightarrow du=-dx$

$x=\alpha\Rightarrow u_2=\alpha-\alpha=0$

$x=0\Rightarrow u_1=\alpha-0=\alpha$

$u=\alpha-x\Rightarrow x=\alpha-u$

β. $\int_0^2 x^{\nu}(2-x)^{\kappa}\,dx = \int_0^2 x^{\kappa}(2-x)^{\nu}\,dx \quad ,\kappa,\nu \in \mathbb{N}^*$

Από πριν είναι: $\int_0^{\alpha} f(x)g(\alpha-x)dx = \int_0^{\alpha} f(\alpha-x)g(x)dx$

Για α = 2 : $\int_0^2 f(x)g(2-x)dx = \int_0^2 f(2-x)g(x)dx$ (1)

Αν $f(x)=x^{\nu}$, $g(x)=x^{\kappa}$,τότε η σχέση (1) γίνεται: $\int_0^2 x^{\nu}(2-x)^{\kappa}\,dx = \int_0^2 x^{\kappa}(2-x)^{\nu}\,dx$

14.54

Αν η συνάρτηση f είναι συνεχής στο [α,β] και f(x) + f(α+β-x)=c για κάθε x∈[α,β] , να αποδείξετε ότι:

$\int_{\alpha}^{\beta} f(x)dx = (\beta-\alpha) f\left(\frac{\alpha+\beta}{2}\right) = \frac{\beta-\alpha}{2}\left[f(\alpha)+f(\beta)\right]$ (Εξετάσεις 1996)

$f(x)+f(\alpha+\beta-x)=c \Rightarrow \int_{\alpha}^{\beta}\left(f(x)+f(\alpha+\beta-x)\right)dx = \int_{\alpha}^{\beta} c dx \Rightarrow$

$\int_{\alpha}^{\beta} f(x)dx + \int_{\alpha}^{\beta} f(\alpha+\beta-x)dx = c\cdot(\beta-\alpha) \Rightarrow$

$\Theta\acute{\epsilon}\tau\omega\ u=\alpha+\beta-x \Rightarrow du=-dx$

$x=\beta \Rightarrow u_2=\alpha+\beta-\beta=\alpha$

$x=\alpha \Rightarrow u_1=\alpha+\beta-\alpha=\beta$

$\int_{\alpha}^{\beta} f(x)dx - \int_{\beta}^{\alpha} f(u)du = c\cdot(\beta-\alpha) \Rightarrow \int_{\alpha}^{\beta} f(x)dx + \int_{\alpha}^{\beta} f(u)du = c\cdot(\beta-\alpha) \Rightarrow$

$\int_{\alpha}^{\beta} f(x)dx + \int_{\alpha}^{\beta} f(x)dx = c\cdot(\beta-\alpha) \Rightarrow 2\int_{\alpha}^{\beta} f(x)dx = c\cdot(\beta-\alpha) \Rightarrow$

$\int_{\alpha}^{\beta} f(x)dx = \frac{c\cdot(\beta-\alpha)}{2} \overset{*}{\Rightarrow} \int_{\alpha}^{\beta} f(x)dx = (\beta-\alpha) f\left(\frac{\alpha+\beta}{2}\right) \overset{**}{=} \frac{\beta-\alpha}{2}\left[f(\alpha)+f(\beta)\right]$

$^*H\ \sigma\chi\acute{\epsilon}\sigma\eta\ f(x)+f(\alpha+\beta-x)=c\ \ \gamma\iota\alpha\ \ x=\frac{\alpha+\beta}{2}\ \ \gamma\acute{\iota}\nu\epsilon\tau\alpha\iota:$

$f\left(\frac{\alpha+\beta}{2}\right)+f\left(\alpha+\beta-\frac{\alpha+\beta}{2}\right)=c \Rightarrow 2f\left(\frac{\alpha+\beta}{2}\right)=c \Rightarrow f\left(\frac{\alpha+\beta}{2}\right)=\frac{c}{2}\,*$

$^{**}H\ \sigma\chi\acute{\epsilon}\sigma\eta\ f(x)+f(\alpha+\beta-x)=c\ \ \gamma\iota\alpha\ \ x=\alpha\ \ \gamma\acute{\iota}\nu\epsilon\tau\alpha\iota:$

$f(\alpha)+f(\beta)=c$

14.55

Αν η συνάρτηση f είναι συνεχής στο [-1,1] και $f(x+y)=f\left(x\right)+f\left(y\right)+3xy$, $x,y\in R$, να αποδείξετε ότι: $\int_{-1}^{1} f\left(x\right)dx=1$

Η $\sigma\chi\acute{\varepsilon}\sigma\eta$ $f(x+y)=f\left(x\right)+f\left(y\right)+3xy$ $\gamma\iota\alpha$ $x=y=0$ $\gamma\acute{\iota}\nu\varepsilon\tau\alpha\iota$:

$f(0)=f\left(0\right)+f\left(0\right)+0\Rightarrow f(0)=0$

Η $\sigma\chi\acute{\varepsilon}\sigma\eta$ $f(x+y)=f\left(x\right)+f\left(y\right)+3xy$ $\gamma\iota\alpha$ $y=-x$ $\gamma\acute{\iota}\nu\varepsilon\tau\alpha\iota$:

$f(x-x)=f\left(x\right)+f\left(-x\right)+3x\left(-x\right)\Rightarrow f(0)=f\left(x\right)+f\left(-x\right)-3x^2\Rightarrow$

$0=f\left(x\right)+f\left(-x\right)-3x^2\Rightarrow f\left(x\right)+f\left(-x\right)=3x^2$

$f\left(x\right)+f\left(-x\right)=3x^2\Rightarrow \int_{-1}^{1}\left(f\left(x\right)+f\left(-x\right)\right)dx=\int_{-1}^{1}3x^2dx\Rightarrow$

$\int_{-1}^{1} f\left(x\right)dx+\int_{-1}^{1} f\left(-x\right)d=\left[x^3\right]_{-1}^{1}\Rightarrow$

Θέτω $u=-x\Rightarrow du=-dx$

$x=1\Rightarrow u_2=-1$

$x=-1\Rightarrow u_1=1$

$\int_{-1}^{1} f\left(x\right)dx+\int_{1}^{-1} f\left(x\right)dx=1-\left(-1\right)\Rightarrow \int_{1}^{-1} f\left(x\right)dx=1$

14.56

α. Αν η συνάρτηση f είναι συνεχής στο R και f(x)≠0 για κάθε x∈R , να αποδείξετε ότι:

$$\int_{\alpha}^{\beta}\frac{f(x-\alpha)}{f(x-\alpha)+f(\beta-x)}dx=\frac{\beta-\alpha}{2}$$

$\Theta\acute{\varepsilon}\tau\omega\ x=\alpha+\beta-u\Rightarrow dx=-du$

$x=\beta\Rightarrow\alpha+\beta-u_2=\beta\Rightarrow u_2=\alpha$

$x=\alpha\Rightarrow\alpha+\beta-u_1=\alpha\Rightarrow u_1=\beta$

$$O\pi\acute{o}\tau\varepsilon:\quad \mathrm{I}=\int_{\alpha}^{\beta}\frac{f(x-\alpha)}{f(x-\alpha)+f(\beta-x)}dx=-\int_{\beta}^{\alpha}\frac{f(\alpha+\beta-u-\alpha)}{f(\alpha+\beta-u-\alpha)+f\big(\beta-(\alpha+\beta-u)\big)}du\Rightarrow$$

$$\mathrm{I}=\int_{\alpha}^{\beta}\frac{f(\beta-u)}{f(\beta-u)+f(u-\alpha)}du\Rightarrow\mathrm{I}=\int_{\alpha}^{\beta}\frac{f(\beta-x)}{f(\beta-x)+f(x-\alpha)}dx$$

$\acute{E}\chi o\upsilon\mu\varepsilon:$

$$\left|\begin{array}{l}\mathrm{I}=\int_{\alpha}^{\beta}\frac{f(x-\alpha)}{f(x-\alpha)+f(\beta-x)}dx\\ \mathrm{I}=\int_{\alpha}^{\beta}\frac{f(\beta-x)}{f(\beta-x)+f(x-\alpha)}dx\end{array}\right.\overset{+}{\Rightarrow}2\mathrm{I}=\int_{\alpha}^{\beta}\frac{f(x-\alpha)+f(\beta-x)}{f(x-\alpha)+f(\beta-x)}dx\Rightarrow2\mathrm{I}=\int_{\alpha}^{\beta}1dx\Rightarrow\mathrm{I}=\frac{\beta-\alpha}{2}$$

β. Αν οι συναρτήσεις f και g είναι συνεχείς στο διάστημα [0,α] και f(x) = f(α-x) , g(x) +g(α-x) = β για κάθε x∈[0,α] να αποδείξετε ότι: $\int_0^{\alpha}f(x)g(x)dx=\frac{\beta}{2}\int_0^{\alpha}f(x)dx$

$\mathrm{I}=\int_0^{\alpha}f(x)g(x)dx=$

$\Theta\acute{\varepsilon}\tau\omega\ x=\alpha-u\Rightarrow dx=-du$

$x=\alpha\Rightarrow\alpha-u_2=\alpha\Rightarrow u_2=0$

$x=0\Rightarrow\alpha-u_1=0\Rightarrow u_1=\alpha$

$\mathrm{I}=\int_0^{\alpha}f(x)g(x)dx=-\int_{\alpha}^{0}f(\alpha-u)g(\alpha-u)du=\int_0^{\alpha}f(u)\big(\beta-g(u)\big)du$

$\mathrm{I}=\int_0^{\alpha}\beta f(u)du-\int_0^{\alpha}f(u)g(u)du$

$\mathrm{I}=\beta\int_0^{\alpha}f(u)du-\mathrm{I}$

$\mathrm{I}=\frac{\beta}{2}\int_0^{\alpha}f(x)dx$

14.57

α. Αν η συνάρτηση f είναι συνεχής στο [-α,α] , f(x) ≠ -1 και f(x)f(-x) = 1 για κάθε x∈[-α,α] να υπολογίσετε το ολοκλήρωμα: $\mathrm{I}=\int_{-\alpha}^{\alpha}\frac{x^{2\nu}}{1+f(x)}dx$

$$\mathrm{I}=\int_{-\alpha}^{\alpha}\frac{x^{2\nu}}{1+f(x)}dx$$

$$\Theta\acute{\varepsilon}\tau\omega\ x=-u\Rightarrow dx=-du$$
$$x=\alpha\Rightarrow u_2=-\alpha$$
$$x=-\alpha\Rightarrow u_1=\alpha$$

$$\mathrm{I}=-\int_{\alpha}^{-\alpha}\frac{(-u)^{2\nu}}{1+f(-u)}du\Rightarrow \mathrm{I}=\int_{-\alpha}^{\alpha}\frac{u^{2\nu}}{1+\frac{1}{f(u)}}du\Rightarrow \mathrm{I}=\int_{-\alpha}^{\alpha}\frac{u^{2\nu}}{\frac{f(u)+1}{f(u)}}du\Rightarrow$$

$$\mathrm{I}=\int_{-\alpha}^{\alpha}\frac{f(u)\cdot u^{2\nu}}{f(u)+1}du\Rightarrow \mathrm{I}=\int_{-\alpha}^{\alpha}\frac{f(x)\cdot x^{2\nu}}{f(x)+1}dx$$

Έχουμε :

$$\left|\begin{array}{l}\mathrm{I}=\int_{-\alpha}^{\alpha}\frac{x^{2\nu}}{1+f(x)}dx\\ \mathrm{I}=\int_{-\alpha}^{\alpha}\frac{f(x)\cdot x^{2\nu}}{f(x)+1}dx\end{array}\right.\overset{+}{\Rightarrow}2\mathrm{I}=\int_{-\alpha}^{\alpha}\frac{x^{2\nu}+f(x)\cdot x^{2\nu}}{1+f(x)}dx\Rightarrow 2\mathrm{I}=\int_{-\alpha}^{\alpha}\frac{(1+f(x))\cdot x^{2\nu}}{1+f(x)}dx\Rightarrow$$

$$\Rightarrow 2\mathrm{I}=\int_{-\alpha}^{\alpha}x^{2\nu}dx\Rightarrow 2\mathrm{I}=\frac{1}{2\nu+1}\left[x^{2\nu+1}\right]_{-\alpha}^{\alpha}\Rightarrow 2\mathrm{I}=\frac{1}{2\nu+1}\left(\alpha^{2\nu+1}-(-\alpha)^{2\nu+1}\right)\Rightarrow$$

$$2\mathrm{I}=\frac{1}{2\nu+1}\left(\alpha^{2\nu+1}+\alpha^{2\nu+1}\right)\Rightarrow \mathrm{I}=\frac{\alpha^{2\nu+1}}{2\nu+1}$$

β. Αν η συνάρτηση f είναι συνεχής στο [α,β] και ισχύει $f(\alpha+\beta-x)=f(x)$ $,x\in R$, να αποδείξετε ότι: $\mathrm{I}=\int_{\alpha}^{\beta}xf(x)dx=\frac{\alpha+\beta}{2}\int_{\alpha}^{\beta}f(x)dx$

$\mathrm{I}=\int_{\alpha}^{\beta}xf(x)dx=$

Θέτω $x=\alpha+\beta-u\Rightarrow dx=-du$

$x=\beta\Rightarrow u_2=\alpha$

$x=\alpha\Rightarrow u_1=\beta$

$$\mathrm{I}=\int_{\alpha}^{\beta}xf(x)dx=\int_{\alpha}^{\beta}(\alpha+\beta-u)f(u)du=\int_{\alpha}^{\beta}(\alpha+\beta)f(u)du-\int_{\alpha}^{\beta}uf(u)du\Rightarrow$$

$$\mathrm{I}=\int_{\alpha}^{\beta}(\alpha+\beta)f(u)du-\mathrm{I}\Rightarrow$$

$$2\mathrm{I}=(\alpha+\beta)\int_{\alpha}^{\beta}f(x)dx\Rightarrow\mathrm{I}=\frac{\alpha+\beta}{2}\int_{\alpha}^{\beta}f(x)dx$$

14.58

Δίνεται η συνάρτηση $f(x)=\frac{x^2+1}{2^x+1}$.

α. Να αποδείξετε ότι $f(x)+f(-x)=x^2+1$ $,x\in R$

$$f(x)+f(-x)=\frac{x^2+1}{2^x+1}+\frac{(-x)^2+1}{2^{-x}+1}=\frac{x^2+1}{2^x+1}+\frac{x^2+1}{\frac{1}{2^x}+1}=\frac{x^2+1}{2^x+1}+\frac{x^2+1}{\frac{1+2^x}{2^x}}=$$

$$=\frac{x^2+1}{2^x+1}+\frac{2^x(x^2+1)}{2^x+1}=\frac{(x^2+1)+2^x(x^2+1)}{2^x+1}=x^2+1$$

β. Να υπολογίσετε το ολοκλήρωμα $\int_{-\alpha}^{\alpha}f(x)dx$

$$f(x)+f(-x)=x^2+1\Rightarrow\int_{-\alpha}^{\alpha}(f(x)+f(-x))dx=\int_{-\alpha}^{\alpha}(x^2+1)dx\Rightarrow$$

$$\int_{-\alpha}^{\alpha}f(x)dx+\int_{-\alpha}^{\alpha}f(-x)dx=\left[\frac{x^3}{3}+x\right]_{-\alpha}^{\alpha}\Rightarrow$$

$$\int_{-\alpha}^{\alpha}f(x)dx+\int_{-\alpha}^{\alpha}f(x)dx=\frac{\alpha^3}{3}+\alpha-\left(\frac{(-\alpha)^3}{3}+(-\alpha)\right)\Rightarrow$$

$$2\int_{-\alpha}^{\alpha}f(x)dx=\frac{\alpha^3}{3}+\alpha-\left(-\frac{\alpha^3}{3}-\alpha\right)\Rightarrow$$

$$2\int_{-\alpha}^{\alpha}f(x)dx=\frac{2\alpha^3}{3}+2\alpha\Rightarrow\int_{-\alpha}^{\alpha}f(x)dx=\frac{\alpha^3}{3}+\alpha$$

14.59

$$A=\int_0^2\frac{1}{1+e^{\sqrt{2-x}-\sqrt{x}}}dx=$$

Θέτω $x=2-u\Rightarrow dx=-du$

$x=2\Rightarrow u_2=0$

$x=0\Rightarrow u_1=2$

$$A=-\int_2^0\frac{1}{1+e^{\sqrt{u}-\sqrt{2-u}}}du=\int_0^2\frac{1}{1+e^{\sqrt{x}-\sqrt{2-x}}}dx=\int_0^2\frac{1}{1+\frac{1}{e^{\sqrt{2-x}-\sqrt{x}}}}dx=\int_0^2\frac{1}{\frac{e^{\sqrt{2-x}-\sqrt{x}}+1}{e^{\sqrt{2-x}-\sqrt{x}}}}dx=$$

$$=\int_0^2\frac{e^{\sqrt{2-x}-\sqrt{x}}}{e^{\sqrt{2-x}-\sqrt{x}}+1}dx$$

Έχουμε :

$$\left|\begin{array}{l} A=\int_0^2\frac{1}{1+e^{\sqrt{2-x}-\sqrt{x}}}dx \\ A=\int_0^2\frac{e^{\sqrt{2-x}-\sqrt{x}}}{e^{\sqrt{2-x}-\sqrt{x}}+1}dx \end{array}\right.\overset{+}{\Rightarrow}2A=\int_0^2 1dx\Rightarrow 2A=2\Rightarrow A=1$$

$$\mathrm{B}=\int_0^2 \ln\frac{e^x+e^2}{e^x+1}dx=$$

Θέτω $x=2-u \Rightarrow dx=-du$

$x=2 \Rightarrow u_2=0$

$x=0 \Rightarrow u_1=2$

$$\mathrm{B}=-\int_2^0 \ln\frac{e^{2-u}+e^2}{e^{2-u}+1}du=\int_0^2 \ln\frac{\frac{e^2}{e^u}+e^2}{\frac{e^2}{e^u}+1}du=\int_0^2 \ln\frac{\frac{e^2+e^2e^u}{e^u}}{\frac{e^2+e^u}{e^u}}du=\int_0^2 \ln\frac{e^2+e^2e^u}{e^2+e^u}du=$$

$$=\int_0^2 \ln\frac{e^2\left(1+e^u\right)}{e^2+e^u}du=\int_0^2 \ln\frac{e^2\left(1+e^x\right)}{e^2+e^x}dx=$$

Έχουμε :

$$\left|\begin{array}{l}\mathrm{B}=\int_0^2 \ln\frac{e^x+e^2}{e^x+1}dx \\ \mathrm{B}=\int_0^2 \ln\frac{e^2\left(1+e^x\right)}{e^2+e^x}dx\end{array}\right. \overset{+}{\Rightarrow} 2\mathrm{B}=\int_0^2 \ln\frac{e^x+e^2}{e^x+1}dx+\int_0^2 \ln\frac{e^2\left(1+e^x\right)}{e^2+e^x}dx \Rightarrow$$

$$2\mathrm{B}=\int_0^2\left(\ln\frac{e^x+e^2}{e^x+1}+\ln\frac{e^2\left(1+e^x\right)}{e^2+e^x}\right)dx \Rightarrow$$

$$2\mathrm{B}=\int_0^2 \ln\left(\frac{e^x+e^2}{e^x+1}\cdot\frac{e^2\left(1+e^x\right)}{e^2+e^x}\right)dx \Rightarrow 2\mathrm{B}=\int_0^2 2dx \Rightarrow \mathrm{B}=2$$

14.60

$$\mathrm{A}=\int_0^{\frac{\pi}{2}}\frac{\sigma\upsilon\nu^7 x}{\eta\mu^7 x+\sigma\upsilon\nu^7 x}dx=$$

$$\Theta\acute{\varepsilon}\tau\omega\ \ x=\frac{\pi}{2}-u\Rightarrow dx=-du$$

$$x=\frac{\pi}{2}\Rightarrow u_2=0$$

$$x=0\Rightarrow u_1=\frac{\pi}{2}$$

$$\mathrm{A}=\int_0^{\frac{\pi}{2}}\frac{\sigma\upsilon\nu^7\left(\frac{\pi}{2}-u\right)}{\eta\mu^7\left(\frac{\pi}{2}-u\right)+\sigma\upsilon\nu^7\left(\frac{\pi}{2}-u\right)}du=\int_0^{\frac{\pi}{2}}\frac{\eta\mu^7 u}{\sigma\upsilon\nu^7 u+\eta\mu^7 u}du=\int_0^{\frac{\pi}{2}}\frac{\eta\mu^7 x}{\sigma\upsilon\nu^7 x+\eta\mu^7 x}dx$$

$$\acute{E}\chi o\upsilon\mu\varepsilon:$$

$$\left|\begin{array}{l}\mathrm{A}=\int_0^{\frac{\pi}{2}}\frac{\sigma\upsilon\nu^7 x}{\eta\mu^7 x+\sigma\upsilon\nu^7 x}dx\\ \mathrm{A}=\int_0^{\frac{\pi}{2}}\frac{\eta\mu^7 x}{\eta\mu^7 x+\sigma\upsilon\nu^7 x}dx\end{array}\right.\overset{+}{\Rightarrow}2\mathrm{A}=\int_0^{\frac{\pi}{2}}1dx\Rightarrow 2\mathrm{A}=\frac{\pi}{2}\Rightarrow \mathrm{A}=\frac{\pi}{4}$$

$$B=\int_0^{\frac{\pi}{2}}\frac{\sqrt[3]{\eta\mu x}}{\sqrt[3]{\eta\mu x}+\sqrt[3]{\sigma\upsilon\nu x}}dx=$$

$\Theta \acute{\varepsilon} \tau \omega\ x=\frac{\pi}{2}-u \Rightarrow dx=-du$

$x=\frac{\pi}{2}\Rightarrow u_2=0$

$x=0\Rightarrow u_1=\frac{\pi}{2}$

$$B=\int_0^{\frac{\pi}{2}}\frac{\sqrt[3]{\eta\mu\left(\frac{\pi}{2}-u\right)}}{\sqrt[3]{\eta\mu\left(\frac{\pi}{2}-u\right)}+\sqrt[3]{\sigma\upsilon\nu\left(\frac{\pi}{2}-u\right)}}du=\int_0^{\frac{\pi}{2}}\frac{\sqrt[3]{\sigma\upsilon\nu x}}{\sqrt[3]{\sigma\upsilon\nu x}+\sqrt[3]{\eta\mu x}}dx=$$

Έχουμε :

$$\left|\begin{array}{l}B=\int_0^{\frac{\pi}{2}}\frac{\sqrt[3]{\eta\mu x}}{\sqrt[3]{\eta\mu x}+\sqrt[3]{\sigma\upsilon\nu x}}dx\\ B=\int_0^{\frac{\pi}{2}}\frac{\sqrt[3]{\sigma\upsilon\nu x}}{\sqrt[3]{\eta\mu x}+\sqrt[3]{\sigma\upsilon\nu x}}dx\end{array}\right.\overset{+}{\Rightarrow}2B=\int_0^{\frac{\pi}{2}}1dx\Rightarrow 2B=\frac{\pi}{2}\Rightarrow B=\frac{\pi}{4}$$

14.61

Έστω f άρτια συνάρτηση και συνεχής στο [-2,2]. Αν F είναι μια παράγουσα της f και F(-2)=F(2) – 2008 , να βρείτε το ολοκλήρωμα $I=\int_0^2 f(x)dx$.

$$\int_{-2}^2 f(x)dx=\int_{-2}^0 f(x)dx+\int_0^2 f(x)dx\overset{*}{=}2\int_0^2 f(x)dx=2I\Rightarrow I=\frac{1}{2}\int_{-2}^2 f(x)dx\Rightarrow$$

$$I=\frac{1}{2}\left[F(x)\right]_{-2}^2=\frac{1}{2}\left(F(2)-F(-2)\right)=\frac{1}{2}\cdot 2008=1004$$

$$*\int_{-2}^0 f(x)dx=-\int_2^0 f(-u)du\overset{\acute{\alpha}\rho\tau\iota\alpha}{=}\int_0^2 f(u)du$$

$\Theta \acute{\varepsilon} \tau \omega\ x=-u\Rightarrow dx=-du$

$x=0\Rightarrow u_2=0$

$x=-2\Rightarrow u_1=2$

14.62

Αν f(x+T) = f(x) για κάθε x∈R , να αποδείξετε ότι: $\int_0^{3T} f(x)dx = 3\int_0^{3T} f(x)dx$

$\int_0^{3T} f(x)dx = \int_0^{T} f(x)dx + \int_T^{2T} f(x)dx + \int_{2T}^{3T} f(x)dx =$

$*\int_T^{2T} f(x)dx = \int_0^{T} f(u+T)du = \int_0^{T} f(u)du$

Θέτω $x = u + T \Rightarrow dx = du$

$x = 2T \Rightarrow u_2 = T$

$x = T \Rightarrow u_1 = 0$

$**\int_{2T}^{3T} f(x)dx = \int_T^{2T} f(u+T)du = \int_T^{2T} f(u)du \overset{*}{=} \int_0^{T} f(u)du$

Θέτω $x = u + T \Rightarrow dx = du$

$x = 3T \Rightarrow u_2 = 2T$

$x = 2T \Rightarrow u_1 = T$

Επομένως $\int_0^{3T} f(x)dx = \int_0^{T} f(x)dx + \int_0^{T} f(x)dx + \int_0^{T} f(x)dx = 3\int_0^{T} f(x)dx$

14.63

α. Έστω f άρτια συνάρτηση και συνεχής στο [-α,α].Να αποδείξετε ότι $\int_{-\alpha}^{\alpha}\frac{f(x)}{e^x+1}dx=\int_0^{\alpha}f(x)dx$

$$\mathrm{I}=\int_{-\alpha}^{\alpha}\frac{f(x)}{e^x+1}dx=$$

Θέτω $x=-u\Rightarrow dx=-du$

$x=\alpha\Rightarrow u_2=-\alpha$

$x=-\alpha\Rightarrow u_1=\alpha$

$$\mathrm{I}=\int_{-\alpha}^{\alpha}\frac{f(-u)}{e^{-u}+1}du\overset{\acute{\alpha}\rho\tau\iota\alpha}{=}\int_{-\alpha}^{\alpha}\frac{f(u)}{\frac{1}{e^u}+1}du=\int_{-\alpha}^{\alpha}\frac{f(u)}{\frac{1+e^u}{e^u}}du=\int_{-\alpha}^{\alpha}\frac{e^u f(u)}{1+e^u}du=\int_{-\alpha}^{\alpha}\frac{e^x f(x)}{1+e^x}dx$$

$\acute{E}\chi o\upsilon\mu\varepsilon:$

$$\left|\begin{aligned}\mathrm{I}&=\int_{-\alpha}^{\alpha}\frac{f(x)}{e^x+1}dx\\ \mathrm{I}&=\int_{-\alpha}^{\alpha}\frac{e^x f(x)}{e^x+1}dx\end{aligned}\right.\overset{+}{\Rightarrow}2\mathrm{I}=\int_{-\alpha}^{\alpha}\left(\frac{f(x)}{e^x+1}+\frac{e^x f(x)}{e^x+1}\right)dx\Rightarrow 2\mathrm{I}=\int_{-\alpha}^{\alpha}f(x)dx\Rightarrow$$

$$\overset{\acute{\alpha}\rho\tau\iota\alpha}{\Rightarrow}2\mathrm{I}=2\int_0^{\alpha}f(x)dx\Rightarrow\mathrm{I}=\int_0^{\alpha}f(x)dx$$

β. Αν α > 1 και f, g συνεχείς στο [-1,1] με f άρτια και g περιττή, να αποδείξετε ότι

$$\int_{-1}^{1}\frac{f(x)}{\alpha^{g(x)}+1}dx=\int_{0}^{1}f(x)dx$$

$$\mathrm{I}=\int_{-1}^{1}\frac{f(x)}{\alpha^{g(x)}+1}dx=$$

Θέτω $x=-u\Rightarrow dx=-du$

$x=1\Rightarrow u_2=-1$

$x=-1\Rightarrow u_1=1$

$$\mathrm{I}=\int_{-1}^{1}\frac{f(-u)}{\alpha^{g(-u)}+1}du\overset{\substack{f\,άρτια\\ g\,περιττή}}{=}\int_{-1}^{1}\frac{f(u)}{\alpha^{-g(u)}+1}du=\int_{-1}^{1}\frac{f(u)}{\frac{1}{\alpha^{g(u)}}+1}du=\int_{-1}^{1}\frac{f(u)}{\frac{1+\alpha^{g(u)}}{\alpha^{g(u)}}}du=\int_{-1}^{1}\frac{\alpha^{g(u)}f(u)}{1+\alpha^{g(u)}}du$$

Έχουμε :

$$\left|\begin{array}{l}\mathrm{I}=\int_{-1}^{1}\frac{f(x)}{\alpha^{g(x)}+1}dx\\ \mathrm{I}=\int_{-1}^{1}\frac{\alpha^{g(x)}\cdot f(x)}{\alpha^{g(x)}+1}dx\end{array}\right.\overset{+}{\Rightarrow}2\mathrm{I}=\int_{-1}^{1}\left(\frac{f(x)}{\alpha^{g(x)}+1}+\frac{\alpha^{g(x)}f(x)}{\alpha^{g(x)}+1}\right)dx\Rightarrow 2\mathrm{I}=\int_{-1}^{1}f(x)dx\Rightarrow$$

$$\overset{άρτια}{\Rightarrow}2\mathrm{I}=2\int_{0}^{1}f(x)dx\Rightarrow \mathrm{I}=\int_{0}^{1}f(x)dx$$

14.64

Αν η f είναι συνεχής στο R και : $f(x)+f(x+1)+f(x+2)=0 \quad ,x\in R$, να αποδείξετε ότι:

α. Η f είναι περιοδική με περίοδο Τ=3.

$f(x)+f(x+1)+f(x+2)=0 \quad (1) \quad ,x\in R$

$για\ x\ το\ x+1:(1)\Rightarrow f(x+1)+f(x+2)+f(x+3)=0\ \ (2)$

$Αφαιρώ\ την\ (1)\ από\ την\ (2)\Rightarrow f(x+3)-f(x)=0\Rightarrow f(x)=f(x+3)$

$Άρα\ η\ f\ είναι\ περιοδική\ με\ \mathrm{T}=3$

β. $\int_{-1}^{1} f(x)\,dx = \int_{0}^{2} f(x+2)\,dx$

$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} f(x+3)\,dx =$

Θέτω $x+3 = u+2 \Rightarrow dx = du$

$x = 1 \Rightarrow u_2 = 2$

$x = -1 \Rightarrow u_1 = 0$

$\int_{-1}^{1} f(x+3)\,dx = \int_{0}^{2} f(u+2)\,du = \int_{0}^{2} f(x+2)\,dx$

14.65

Αν η f είναι συνεχής στο R και περιοδική , να αποδείξετε ότι:

$\boxed{\int_{\alpha}^{\alpha+T} f(x)dx = \int_{\beta}^{\beta+T} f(x)dx \quad ,\alpha,\beta \in R}$

$\int_{\alpha}^{\alpha+T} f(x)dx = \int_{\alpha}^{T} f(x)dx + \int_{T}^{\alpha+T} f(x)dx^* =$

*Θέτω $x = u + T \Rightarrow dx = du$

$x = \alpha + T \Rightarrow u_2 = \alpha$

$x = T \Rightarrow u_1 = 0$

$\int_{T}^{\alpha+T} f(x)dx = \int_{0}^{\alpha} f(u+T)du \overset{\pi\varepsilon\rho\iota o\delta\iota\kappa\acute{\eta}}{=} \int_{0}^{\alpha} f(u)du^*$

Επομένως : $\int_{\alpha}^{\alpha+T} f(x)dx = \int_{\alpha}^{T} f(x)dx + \int_{T}^{\alpha+T} f(x)dx$

$\overset{*}{=} \int_{\alpha}^{T} f(x)dx + \int_{0}^{\alpha} f(x)dx$

$= \int_{0}^{T} f(x)dx$

Ομοίως , $\int_{\beta}^{\beta+T} f(x)dx = \int_{0}^{T} f(x)dx$

Άρα : $\int_{\alpha}^{\alpha+T} f(x)dx = \int_{\beta}^{\beta+T} f(x)dx$

14.66

Δίνεται η συνάρτηση $f(x) = x^3+x+1$

α. Να δείξετε ότι η f αντιστρέφεται και να βρείτε το πεδίο ορισμού της f^{-1}.

$f'(x) = 3x^2 + 1 > 0 \Rightarrow f$ *γν.αύξουσα στο R*

$f(A) = \left(\lim_{x\to-\infty} f(x), \lim_{x\to+\infty} f(x)\right) = R$

Το πεδίο ορισμού της f^{-1} είναι το R.

β. Να υπολογίσετε το ολοκλήρωμα $I=\int_{-1}^{1} f^{-1}(x)dx$

$f(x)=x^3+x+1$, $f'(x)=3x^2+1$

$I=\int_{-1}^{1} f^{-1}(x)dx$

Θέτω $y=f^{-1}(x)\Rightarrow x=f(y)\Rightarrow dx=f'(y)dy\Rightarrow dx=(3y^2+1)dy$

$x=1\Rightarrow f(y)=1\Rightarrow y^3+y+1=1\Rightarrow y=0$

$x=-1\Rightarrow f(y)=-1\Rightarrow y^3+y+1=-1\Rightarrow y^3+y+2=0\Rightarrow y=-1$

$I=\int_{-1}^{1} f^{-1}(x)dx=\int_{-1}^{0} y(3y^2+1)dy=\int_{-1}^{0}(3y^3+y)dy=\left[\frac{3y^4}{4}+\frac{y^2}{2}\right]_{-1}^{0}$

$I=0-\left(\frac{3}{4}+\frac{1}{2}\right)=-\frac{5}{4}$

14.67

Αν η συνάρτηση f είναι γνησίως μονότονη και έχει συνεχή παράγωγο στο [α,β] , να αποδείξετε ότι: $\int_{\alpha}^{\beta} f(x)dx+\int_{f(\alpha)}^{f(\beta)} f^{-1}(x)dx=\beta f(\beta)-\alpha f(\alpha)$

$\int_{f(\alpha)}^{f(\beta)} f^{-1}(x)dx=$

f γν. μονότονη $\Rightarrow f:1-1$

Θέτω $y=f^{-1}(x)\Rightarrow x=f(y)\Rightarrow dx=f'(y)dy$

$x=f(\beta)\overset{f:1-1}{\Rightarrow} f(y)=f(\beta)\Rightarrow y=\beta$

$x=f(\alpha)\overset{f:1-1}{\Rightarrow} f(y)=f(\alpha)\Rightarrow y=\alpha$

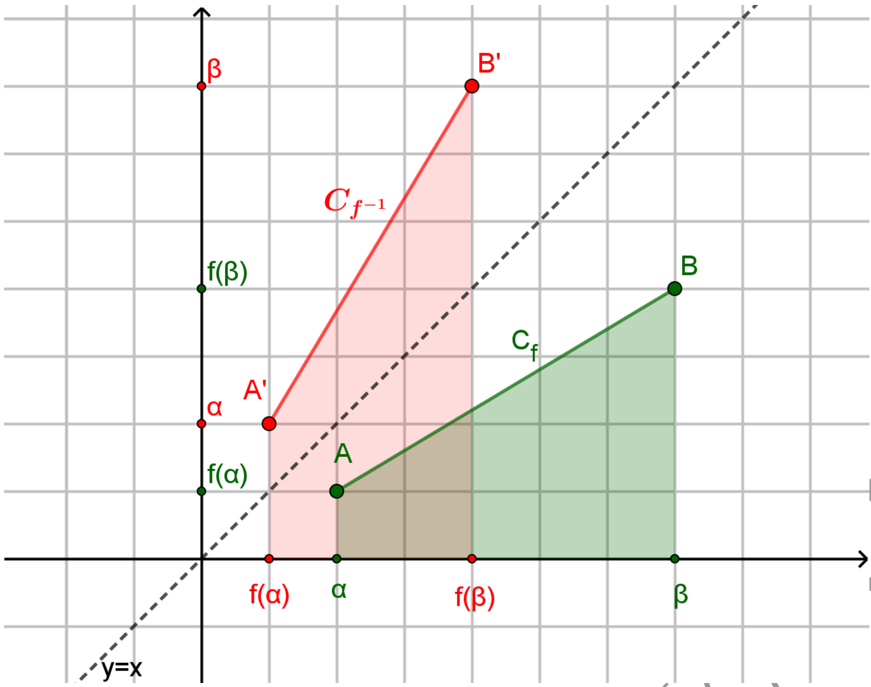
$\int_{f(\alpha)}^{f(\beta)} f^{-1}(x)dx=\int_{\alpha}^{\beta} yf'(y)dy=\left[yf(y)\right]_{\alpha}^{\beta}-\int_{\alpha}^{\beta}(y)'f(y)dy\Rightarrow$

$\int_{f(\alpha)}^{f(\beta)} f^{-1}(x)dx=\beta f(\beta)-\alpha f(\alpha)-\int_{\alpha}^{\beta} f(y)dy\Rightarrow$

$\int_{f(\alpha)}^{f(\beta)} f^{-1}(x)dx=\beta f(\beta)-\alpha f(\alpha)-\int_{\alpha}^{\beta} f(x)dx\Rightarrow$

$\int_{\alpha}^{\beta} f(x)dx+\int_{f(\alpha)}^{f(\beta)} f^{-1}(x)dx=\beta f(\beta)-\alpha f(\alpha)$

Γεωμετρική Ερμηνεία – Απόδειξη



Με βάση το παραπάνω σχήμα:

$$\int_{\alpha}^{\beta} f(x)dx = (\text{Εμβαδόν Πράσινου Τραπεζίου}) = \frac{\left(f(\beta)+f(\alpha)\right)\cdot(\beta-\alpha)}{2}$$

$$\int_{f(\alpha)}^{f(\beta)} f^{-1}(x)dx = (\text{Εμβαδόν Κόκκινου Τραπεζίου}) = \frac{(\beta+\alpha)\cdot\left(f(\beta)-f(\alpha)\right)}{2}$$

Οπότε

$$\int_{\alpha}^{\beta} f(x)dx + \int_{f(\alpha)}^{f(\beta)} f^{-1}(x)dx = \frac{\left(f(\beta)+f(\alpha)\right)\cdot(\beta-\alpha)}{2} + \frac{(\beta+\alpha)\cdot\left(f(\beta)-f(\alpha)\right)}{2} =$$

$$= \ldots\ldots\ldots = \beta f(\beta) - \alpha f(\alpha)$$

14.68

Έστω η συνάρτηση $f(x)=\eta\mu x \quad , x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

α. Να αποδείξετε ότι η f αντιστρέφεται

$$f'(x)=\sigma\upsilon\nu x>0 \quad \gamma\iota\alpha\ \kappa\acute{\alpha}\theta\varepsilon\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

Άρα f γν. αύξουσα ⇒ f αντιστρέφεται

β. Να υπολογίσετε το ολοκλήρωμα $A=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^2(x)dx$

Λύνεται με την βοήθεια τριγωνομετρικών τύπων διπλασίου γωνίας (εκτός ύλης) . $A=\frac{\pi}{2}$

γ. Να αποδείξετε ότι $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^2(x)dx + 2\int_{-1}^{1} f^{-1}(x)dx = \pi$

$$\int_{-1}^{1} xf^{-1}(x)dx =$$

f *γν. μονότονη* $\Rightarrow f:1-1$

Θέτω $y = f^{-1}(x) \Rightarrow x = f(y) \Rightarrow dx = f'(y)\,dy$

$$x = 1 \Rightarrow f(y) = 1 \Rightarrow \eta\mu y = 1 \Rightarrow y = \frac{\pi}{2}$$

$$x = -1 \Rightarrow f(y) = -1 \Rightarrow \eta\mu y = -1 \Rightarrow y = -\frac{\pi}{2}$$

$$\int_{-1}^{1} xf^{-1}(x)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} yf(y)f'(y)\,dy = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y\left(f^2(y)\right)'\,dy \Rightarrow$$

$$\int_{-1}^{1} xf^{-1}(x)dx = \frac{1}{2}\left(\left[yf^2(y)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (y)' f^2(y)\,dy\right) \Rightarrow$$

$$\int_{-1}^{1} xf^{-1}(x)dx = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^2(y)\,dy\right) \Rightarrow$$

$$\int_{-1}^{1} xf^{-1}(x)dx = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2}\right) \Rightarrow \int_{-1}^{1} f^{-1}(x)dx = \frac{\pi}{4}$$

Ε*πομένως* :

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^2(x)dx + 2\int_{-1}^{1} xf^{-1}(x)dx = \frac{\pi}{2} + 2\cdot\frac{\pi}{4} = \pi$$

δ. Να αποδείξετε ότι $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^{\nu}(x)dx+\nu\int_{-1}^{1} x^{\nu-1}f^{-1}(x)dx=\pi$ $\quad ,\nu$ *άρτιος φυσικός* , $\nu\neq 0$

$\int_{-1}^{1} x^{\nu-1}f^{-1}(x)dx=$

f *γν. μονότονη* $\Rightarrow f:1-1$

Θέτω $y=f^{-1}(x)\Rightarrow x=f(y)\Rightarrow dx=f'(y)dy$

$x=1\Rightarrow f(y)=1\Rightarrow \eta\mu y=1\Rightarrow y=\frac{\pi}{2}$

$x=-1\Rightarrow f(y)=-1\Rightarrow \eta\mu y=-1\Rightarrow y=-\frac{\pi}{2}$

$$\int_{-1}^{1} x^{\nu-1}f^{-1}(x)dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} yf^{\nu-1}(y)f'(y)dy=\frac{1}{\nu}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y\left(f^{\nu}(y)\right)'dy\Rightarrow$$

$$\int_{-1}^{1} x^{\nu-1}f^{-1}(x)dx=\frac{1}{\nu}\left(\left[yf^{\nu-1}(y)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(y)'f^{\nu}(y)dy\right)\Rightarrow$$

$$\int_{-1}^{1} x^{\nu-1}f^{-1}(x)dx=\frac{1}{\nu}\left(\frac{\pi}{2}+\frac{\pi}{2}-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^{\nu}(y)dy\right)\Rightarrow$$

$$\int_{-1}^{1} x^{\nu-1}f^{-1}(x)dx=\frac{1}{\nu}\left(\pi-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^{\nu}(x)dx\right)\Rightarrow$$

$$\int_{-1}^{1} x^{\nu-1}f^{-1}(x)dx=\frac{\pi}{\nu}-\frac{1}{\nu}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^{\nu}(x)dx$$

Επ*ομένως* :

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^{\nu}(x)dx+\nu\int_{-1}^{1} x^{\nu-1}f^{-1}(x)dx&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^{\nu}(x)dx+\nu\cdot\left(\frac{\pi}{\nu}-\frac{1}{\nu}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^{\nu}(x)dx\right)\\&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^{\nu}(x)dx+\pi-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^{\nu}(x)dx\\&=\pi\end{aligned}$$

14.69

Δίνεται η συνεχής συνάρτηση f :R→R για την οποία ισχύει : $f(x)+f(-x)=2 \quad ,x\in R$

α. Να αποδείξετε ότι $\int_{-1}^{1} f(-x)\,dx = \int_{-1}^{1} f(x)dx$

$$\int_{-1}^{1} f(-x)\,dx =$$

Θέτω $x=-u \Rightarrow dx=-du$

$x=1 \Rightarrow u_2=-1$

$x=-1 \Rightarrow u_1=1$

$$\int_{-1}^{1} f(-x)\,dx = -\int_{1}^{-1} f(u)\,du = \int_{-1}^{1} f(u)\,du = x$$

β. Να υπολογίσετε το ολοκλήρωμα $\mathrm{A}=\int_{-1}^{1} f(x)dx$

$f(x)+f(-x)=2 \Rightarrow$

$\int_{-1}^{1}\left(f(x)+f(-x)\right)dx = \int_{-1}^{1} 2dx \Rightarrow$

$\int_{-1}^{1} f(x)\,dx + \int_{-1}^{1} f(-x)\,dx = 2\cdot(1+1) \Rightarrow$

$\mathrm{A}+\mathrm{A}=4 \Rightarrow \mathrm{A}=2$

14.70

Μια συνάρτηση f με f(0) = 1 και f(1) = 2 είναι παραγωγίσιμη στο [0,1].
Να βρείτε τα ολοκληρώματα:

α. $\mathrm{A}=\int_{0}^{1} x^2\left(3f(x)+xf'(x)\right)dx$

Αφού η f είναι παραγωγίσιμη, θα είναι και συνεχής στο [0,1].

$$\mathrm{A}=\int_{0}^{1} x^2\left(3f(x)+xf'(x)\right)dx = \int_{0}^{1}\left(3x^2 f(x)+x^3 f'(x)\right)dx =$$

$$=\int_{0}^{1}\left(x^3 f(x)\right)' dx = \left[x^3 f(x)\right]_0^1 = 1\cdot f(1)-0=2$$

β. $\mathrm{B}=\int_{0}^{1}\frac{f'(x)-f(x)}{e^x}dx$

$$\mathrm{B}=\int_{0}^{1}\frac{f'(x)-f(x)}{e^x}dx = \int_{0}^{1}\frac{\left(f'(x)-f(x)\right)\cdot e^x}{e^x\cdot e^x}dx = \int_{0}^{1}\frac{e^x f'(x)-\left(e^x\right)' f(x)}{e^{2x}}dx =$$

$$=\int_{0}^{1}\left(\frac{f(x)}{e^x}\right)' dx = \left[\frac{f(x)}{e^x}\right]_0^1 = \frac{f(1)}{e}-\frac{f(0)}{1} = \frac{2}{e}-1$$

14.71

Να αποδείξετε ότι δεν υπάρχει συνάρτηση f με συνεχή παράγωγο στο διάστημα [1,3] ,
$f'(x) \geq 3x^2 \quad , x \in [1,3] \quad , f(3) = 31 \quad , f(1) = 6$

Έστω ότι υπάρχει τέτοια συνάρτηση. Τότε:

$$f'(x) \geq 3x^2 \Rightarrow \int_1^3 f'(x)dx \geq \int_1^3 3x^2 dx \Rightarrow \int_1^3 f'(x)dx \geq \int_1^3 \left(x^3\right)' dx \Rightarrow$$

$$\left[f(x)\right]_1^3 \geq \left[x^3\right]_1^3 \Rightarrow f(3) - f(1) \geq 27 - 1 \Rightarrow 31 - 6 \geq 26 \Rightarrow 25 \geq 26$$

που είναι ΑΤΟΠΟ.

14.72

Αν $f'(x) = \sqrt{1 + f^2(x)} \quad , \quad x \in R \quad , f(0) + f(1) = 0$, να υπολογίσετε το ολοκλήρωμα
$\mathrm{A} = \int_0^1 f(x)dx$

$$f'(x) = \sqrt{1 + f^2(x)} \Rightarrow \frac{f'(x)}{\sqrt{1 + f^2(x)}} = 1 \Rightarrow \frac{2f'(x)}{2\sqrt{1 + f^2(x)}} = 1 \overset{f(x) \neq 0}{\Rightarrow} \frac{2f'(x)f(x)}{2\sqrt{1 + f^2(x)}} = f(x)$$

$$\mathrm{A} = \int_0^1 f(x)dx = \int_0^1 \frac{2f'(x)f(x)}{2\sqrt{1 + f^2(x)}}dx = \int_0^1 \frac{\left(1 + f^2(x)\right)'}{2\sqrt{1 + f^2(x)}}dx = \int_0^1 \left(\sqrt{1 + f^2(x)}\right)' dx =$$

$$= \left[\sqrt{1 + f^2(x)}\right]_0^1 = \left(\sqrt{1 + f^2(1)}\right) - \left(\sqrt{1 + f^2(0)}\right) = \left(\sqrt{1 + f^2(1)}\right) - \left(\sqrt{1 + \left(-f(1)\right)^2}\right) =$$

$$= \left(\sqrt{1 + f^2(1)}\right) - \left(\sqrt{1 + f^2(1)}\right) = 0$$

14.73

Δίνεται η συνεχής συνάρτηση f :R→R . Να αποδείξετε ότι:

α . $\int_0^{2\alpha} f(x)\,dx = \int_0^{\alpha}\left(f(x)+f(2\alpha-x)\right)dx$

$$\int_0^{2\alpha} f(x)\,dx = \int_0^{\alpha} f(x)dx + \int_{\alpha}^{2\alpha} f(x)dx\,*$$

$$*\int_{\alpha}^{2\alpha} f(x)dx =$$

Θέτω $x = 2\alpha - u \Rightarrow dx = -du$

$x = 2\alpha \Rightarrow u_2 = 0$

$x = \alpha \Rightarrow u_1 = \alpha$

$$\int_{\alpha}^{2\alpha} f(x)dx = -\int_{\alpha}^{0} f(2\alpha-u)du = \int_0^{\alpha} f(2\alpha-u)du = \int_0^{\alpha} f(2\alpha-x)dx$$

Οπότε :

$$\int_0^{2\alpha} f(x)\,dx = \int_0^{\alpha} f(x)dx + \int_{\alpha}^{2\alpha} f(x)dx = \int_0^{\alpha} f(x)dx + \int_0^{\alpha} f(2\alpha-x)dx \Rightarrow$$

$$\int_0^{2\alpha} f(x)\,dx = \int_0^{\alpha}\left(f(x)+f(2\alpha-x)\right)dx$$

β . $\int_0^{\frac{\pi}{2}} f(\eta\mu x)\,dx = \int_0^{\frac{\pi}{2}} f(\sigma\upsilon\nu x)\,dx$

$$\int_0^{\frac{\pi}{2}} f(\eta\mu x)\,dx =$$

Θέτω $x = \frac{\pi}{2} - u \Rightarrow dx = -du$

$x = \frac{\pi}{2} \Rightarrow u_2 = 0$

$x = 0 \Rightarrow u_1 = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} f(\eta\mu x)\,dx = -\int_{\frac{\pi}{2}}^{0} f\left(\eta\mu\left(\frac{\pi}{2}-u\right)\right)du = \int_0^{\frac{\pi}{2}} f(\sigma\upsilon\nu u)\,du = \int_0^{\frac{\pi}{2}} f(\sigma\upsilon\nu x)\,dx$$

γ . $\int_0^{\frac{\pi}{2}} f(\sigma\upsilon\nu x)\,dx = \int_{\frac{\pi}{2}}^{\pi} f(\eta\mu x)\,dx$

$\int_0^{\frac{\pi}{2}} f(\sigma\upsilon\nu x)\,dx =$

Θέτω $x = u - \frac{\pi}{2} \Rightarrow dx = du$

$x = \frac{\pi}{2} \Rightarrow u_2 = \pi$

$x = 0 \Rightarrow u_1 = \frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} f(\sigma\upsilon\nu x)\,dx = \int_{\frac{\pi}{2}}^{\pi} f\left(\sigma\upsilon\nu\left(u - \frac{\pi}{2}\right)\right)du = \int_{\frac{\pi}{2}}^{\pi} f\left(\sigma\upsilon\nu\left(\frac{\pi}{2} - u\right)\right)du =$

$= \int_{\frac{\pi}{2}}^{\pi} f(\eta\mu u)\,du = \int_{\frac{\pi}{2}}^{\pi} f(\eta\mu x)\,dx$

δ . $\int_{-\pi}^{\pi} f(\sigma\upsilon\nu x)\,dx = 2\int_0^{\pi} f(\sigma\upsilon\nu x)\,dx$

$\int_{-\pi}^{\pi} f(\sigma\upsilon\nu x)\,dx = \int_{-\pi}^{0} f(\sigma\upsilon\nu x)\,dx^* + \int_0^{\pi} f(\sigma\upsilon\nu x)\,dx$

*Θέτω $x = -u \Rightarrow dx = -du$

$x = 0 \Rightarrow u_2 = 0$

$x = -\pi \Rightarrow u_1 = \pi$

$\int_{-\pi}^{0} f(\sigma\upsilon\nu x)\,dx = -\int_{\pi}^{0} f(\sigma\upsilon\nu(-u))\,du = \int_0^{\pi} f(\sigma\upsilon\nu u)\,du = \int_0^{\pi} f(\sigma\upsilon\nu x)\,dx =$

Άρα :

$\int_{-\pi}^{\pi} f(\sigma\upsilon\nu x)\,dx = \int_0^{\pi} f(\sigma\upsilon\nu x)\,dx + \int_0^{\pi} f(\sigma\upsilon\nu x)\,dx = 2\int_0^{\pi} f(\sigma\upsilon\nu x)\,dx$

14.74

Να βρείτε την συνάρτηση f :R→R με $f(x)=2x+\int_1^3 f(t)\,dt$, $x\in R$

$f(x)=2x+\int_1^3 f(t)\,dt$, $x\in R$ (1)

$Έστω\ \int_1^3 f(t)\,dt = c\ ,c\in R$

Είναι: $f(x)=2x+c$

$(1)\Rightarrow 2x+c=2x+\int_1^3 (2t+c)\,dt \Rightarrow c=\left[t^2+ct\right]_1^3 \Rightarrow c=(9+3c)-(1+c)\Rightarrow$

$\Rightarrow c=9+3c-1-c\Rightarrow c=-8$

Επομένως: $f(x)=2x-8$

14.75

Αν η συνάρτηση f είναι συνεχής στο διάστημα [0,1] , να αποδείξετε ότι:

α. $\int_{\frac{\pi}{2}}^{\pi} f(\eta\mu x)\,dx=\int_0^{\frac{\pi}{2}} f(\eta\mu x)\,dx$

$\int_{\frac{\pi}{2}}^{\pi} f(\eta\mu x)\,dx=$

Θέτω $x=\pi-u\Rightarrow dx=-du$

$x=\pi\Rightarrow u_2=0$

$x=\frac{\pi}{2}\Rightarrow u_1=\frac{\pi}{2}$

$$\int_{\frac{\pi}{2}}^{\pi} f(\eta\mu x)\,dx=-\int_{\frac{\pi}{2}}^{0} f(\eta\mu(\pi-u))\,du=\int_0^{\frac{\pi}{2}} f(\eta\mu u)\,du=\int_0^{\frac{\pi}{2}} f(\eta\mu x)\,dx$$

β. $\int_0^{\pi} f(\eta\mu x)\,dx=2\int_0^{\frac{\pi}{2}} f(\eta\mu x)\,dx$

$$\int_0^{\pi} f(\eta\mu x)\,dx=\int_0^{\frac{\pi}{2}} f(\eta\mu x)\,dx+\int_{\frac{\pi}{2}}^{\pi} f(\eta\mu x)\,dx\overset{(\alpha)}{=}$$

$$=\int_0^{\frac{\pi}{2}} f(\eta\mu x)\,dx+\int_0^{\frac{\pi}{2}} f(\eta\mu x)\,dx=$$

$$=2\int_0^{\frac{\pi}{2}} f(\eta\mu x)\,dx$$

14.76

Να αποδείξετε ότι:

α. $\int_0^{\pi} \sigma\upsilon\nu^7 x dx = 0$

$I = \int_0^{\pi} \sigma\upsilon\nu^7 x dx =$

Θέτω $x = \pi - u \Rightarrow dx = -du$

$x = \pi \Rightarrow u_2 = 0$

$x = 0 \Rightarrow u_1 = \pi$

$I = \int_0^{\pi} \sigma\upsilon\nu^7 x dx = \int_0^{\pi} \sigma\upsilon\nu^7 (\pi - u) du = -\int_0^{\pi} \sigma\upsilon\nu^7 u du = -I$

Επομένως: $I = -I \Rightarrow 2I = 0 \Rightarrow I = 0$

β. $\int_0^{2\pi} \sigma\upsilon\nu^7 x dx = 0$

$I = \int_0^{2\pi} \sigma\upsilon\nu^7 x dx = \int_0^{\pi} \sigma\upsilon\nu^7 x dx + \int_{\pi}^{2\pi} \sigma\upsilon\nu^7 x dx \overset{(\alpha)}{=} 0 + \int_{\pi}^{2\pi} \sigma\upsilon\nu^7 x dx$

Θέτω $x = \pi + u \Rightarrow dx = du$

$x = 2\pi \Rightarrow u_2 = \pi$

$x = \pi \Rightarrow u_1 = 0$

$I = \int_{\pi}^{2\pi} \sigma\upsilon\nu^7 x dx = \int_{\pi}^{0} \sigma\upsilon\nu^7 (\pi + u) du = -\int_0^{\pi} \sigma\upsilon\nu^7 (\pi + u) du = -\int_0^{\pi} \sigma\upsilon\nu^7 u du \overset{(\alpha)}{=} 0$

γ. $\int_{\frac{1}{3}}^{\frac{2}{3}} \ln\left(\frac{1}{x} - 1\right) dx = 0$

$I = \int_{\frac{1}{3}}^{\frac{2}{3}} \ln\left(\frac{1}{x} - 1\right) dx = \int_{\frac{1}{3}}^{\frac{2}{3}} \ln\left(\frac{1-x}{x}\right) dx =$

*Θέτω $1 - x = u \Rightarrow -dx = du \Rightarrow dx = -du$

$x = \frac{2}{3} \Rightarrow u_2 = \frac{1}{3}$

$x = \frac{1}{3} \Rightarrow u_1 = \frac{2}{3}$

$I = \int_{\frac{1}{3}}^{\frac{2}{3}} \ln\left(\frac{1-x}{x}\right) dx = -\int_{\frac{2}{3}}^{\frac{1}{3}} \ln\left(\frac{u}{1-u}\right) du = \int_{\frac{1}{3}}^{\frac{2}{3}} \ln\left(\frac{u}{1-u}\right) du = \int_{\frac{1}{3}}^{\frac{2}{3}} \ln\left(\frac{1-u}{u}\right)^{-1} du \Rightarrow$

$I = -\int_{\frac{1}{3}}^{\frac{2}{3}} \ln\left(\frac{1-u}{u}\right) du \Rightarrow I = -I \Rightarrow I = 0$

δ. $\boxed{\int_0^{\frac{\pi}{2}} \frac{\eta\mu x - \sigma\upsilon\nu x}{1+\eta\mu x\sigma\upsilon\nu x}dx = 0}$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\eta\mu x - \sigma\upsilon\nu x}{1+\eta\mu x\sigma\upsilon\nu x}dx =$$

Θέτω $x = \frac{\pi}{2} - u \Rightarrow dx = -du$

$x = \frac{\pi}{2} \Rightarrow u_2 = 0$

$x = 0 \Rightarrow u_1 = \frac{\pi}{2}$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\eta\mu x - \sigma\upsilon\nu x}{1+\eta\mu x\sigma\upsilon\nu x}dx = \int_0^{\frac{\pi}{2}} \frac{\eta\mu\left(\frac{\pi}{2}-u\right) - \sigma\upsilon\nu\left(\frac{\pi}{2}-u\right)}{1+\eta\mu\left(\frac{\pi}{2}-u\right)\sigma\upsilon\nu\left(\frac{\pi}{2}-u\right)}du \Rightarrow$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sigma\upsilon\nu u - \eta\mu u}{1+\sigma\upsilon\nu u\eta\mu u}du \Rightarrow \mathrm{I} = -\mathrm{I} \Rightarrow \mathrm{I} = 0$$

14.77

Αν η συνάρτηση f έχει συνεχή δεύτερη παράγωγο στο [0,π] , η γρ. παράσταση της f ΄ διέρχεται από το Α(π,1) και ισχύει $\int_0^{\pi}\left(f(x)+f''(x)\right)\sigma\upsilon\nu x dx = 1$, να εξετάσετε αν η γρ. παράσταση της f ΄ διέρχεται από το Β(0,-2).

Είναι $f'(\pi)=1$. Αρκεί να εξετάσω αν $f'(0)=-2$

$\int_0^{\pi}\left(f(x)+f''(x)\right)\sigma\upsilon\nu x dx = 1 \Leftrightarrow$

$\int_0^{\pi}\left(f(x)\sigma\upsilon\nu x + f''(x)\sigma\upsilon\nu x\right)dx = 1 \Leftrightarrow$

$\int_0^{\pi}\left(f(x)\sigma\upsilon\nu x + f'(x)\eta\mu x - f'(x)\eta\mu x + f''(x)\sigma\upsilon\nu x\right)dx = 1 \Leftrightarrow$

$\int_0^{\pi}\left(f(x)\sigma\upsilon\nu x + f'(x)\eta\mu x\right)dx + \int_0^{\pi}\left(-f'(x)\eta\mu x + f''(x)\sigma\upsilon\nu x\right)dx = 1 \Leftrightarrow$

$\int_0^{\pi}\left(f(x)(\eta\mu x)' + f'(x)\eta\mu x\right)dx + \int_0^{\pi}\left(f'(x)(\sigma\upsilon\nu x)' + \left(f'(x)\right)'\sigma\upsilon\nu x\right)dx = 1 \Leftrightarrow$

$\int_0^{\pi}\left(f(x)\eta\mu x\right)' dx + \int_0^{\pi}\left(f'(x)\sigma\upsilon\nu x\right)' dx = 1 \Leftrightarrow$

$\left[f(x)\eta\mu x\right]_0^{\pi} - \left[f'(x)\sigma\upsilon\nu x\right]_0^{\pi} = 1 \Rightarrow$

$\left(f(\pi)\eta\mu\pi - f(0)\eta\mu 0\right) - \left(f'(\pi)\sigma\upsilon\nu\pi - f'(0)\sigma\upsilon\nu 0\right) = 1 \Rightarrow$

$(0-0) - \left(1\cdot(-1) - f'(0)\cdot 1\right) = 1 \Rightarrow$

$-1 - f'(0) = 1 \Rightarrow f'(0) = -2$

14.78

Αν η συνάρτηση f είναι συνεχής στο R , να αποδείξετε ότι:

α. $\int_{\alpha}^{\beta} f(\alpha+\beta-x)\,dx = \int_{\alpha}^{\beta} f(x)\,dx$

$\int_{\alpha}^{\beta} f(\alpha+\beta-x)\,dx =$

Θέτω $\alpha+\beta-x=u \Rightarrow dx=-du$

$x=\beta \Rightarrow u_2=\alpha$

$x=\alpha \Rightarrow u_1=\beta$

$$\int_{\alpha}^{\beta} f(\alpha+\beta-x)\,dx = -\int_{\beta}^{\alpha} f(u)\,du = \int_{\alpha}^{\beta} f(u)\,du = \int_{\alpha}^{\beta} f(x)\,dx$$

β. $\int_{\alpha+\gamma}^{\beta+\gamma} f(x-\gamma)\,dx = \int_{\alpha}^{\beta} f(x)\,dx$

$\int_{\alpha+\gamma}^{\beta+\gamma} f(x-\gamma)\,dx =$

Θέτω $x-\gamma=u \Rightarrow dx=du$

$x=\beta+\gamma \Rightarrow u_2=\beta$

$x=\alpha+\gamma \Rightarrow u_1=\alpha$

$$\int_{\alpha+\gamma}^{\beta+\gamma} f(x-\gamma)\,dx = \int_{\alpha}^{\beta} f(u)\,du = \int_{\alpha}^{\beta} f(x)\,dx$$

γ. $\int_{\alpha\gamma}^{\beta\gamma} f\left(\frac{x}{\gamma}\right)dx = \gamma\int_{\alpha}^{\beta} f(x)\,dx \quad ,\gamma \neq 0$

$\int_{\alpha\gamma}^{\beta\gamma} f\left(\frac{x}{\gamma}\right)dx =$

Θέτω $\frac{x}{\gamma}=u \Rightarrow \frac{1}{\gamma}dx=du \Rightarrow dx=\gamma du$

$x=\beta\gamma \Rightarrow u_2=\beta$

$x=\alpha\gamma \Rightarrow u_1=\alpha$

$$\int_{\alpha\gamma}^{\beta\gamma} f\left(\frac{x}{\gamma}\right)dx = \int_{\alpha}^{\beta} u\cdot\gamma du = \gamma\int_{\alpha}^{\beta} f(x)\,dx$$

δ. $\boxed{\int_{\frac{\alpha}{\gamma}}^{\frac{\beta}{\gamma}} f(x\gamma)\,dx = \frac{1}{\gamma}\int_{\alpha}^{\beta} f(x)\,dx \quad ,\gamma \neq 0}$

$$\int_{\frac{\alpha}{\gamma}}^{\frac{\beta}{\gamma}} f(x\gamma)\,dx =$$

Θέτω $x\gamma = u \Rightarrow \gamma dx = du \Rightarrow dx = \frac{1}{\gamma}du$

$$x = \frac{\beta}{\gamma} \Rightarrow u_2 = \beta$$

$$x = \frac{\alpha}{\gamma} \Rightarrow u_1 = \alpha$$

$$\int_{\alpha\gamma}^{\beta\gamma} f\left(\frac{x}{\gamma}\right)dx = \int_{\alpha}^{\beta} u \cdot \frac{1}{\gamma}du = \frac{1}{\gamma}\int_{\alpha}^{\beta} f(x)\,dx$$

ε. $\boxed{(\beta-\alpha)\int_0^1 f\left[\alpha+(\beta-\alpha)x\right]dx = \int_{\alpha}^{\beta} f(x)\,dx}$

$$(\beta-\alpha)\int_0^1 f\left[\alpha+(\beta-\alpha)x\right]dx =$$

Θέτω $\alpha+(\beta-\alpha)x = u \Rightarrow (\beta-\alpha)dx = du$

$$x = 1 \Rightarrow u_2 = \beta$$

$$x = 0 \Rightarrow u_1 = \alpha$$

$$(\beta-\alpha)\int_0^1 f\left[\alpha+(\beta-\alpha)x\right]dx = \int_0^1 f\left[\alpha+(\beta-\alpha)x\right](\beta-\alpha)\,dx =$$

$$= \int_{\alpha}^{\beta} f(u)\,du = \int_{\alpha}^{\beta} f(x)\,dx$$

14.79

Έστω συνάρτηση f συνεχής με συνεχή δεύτερη παράγωγο στο R. Αν ισχύουν:

$f'(2)=0$, $f(0)=-2$ $και$ $\int_0^2\left[xf''(x)+3f'(x)\right]dx=4016$, να βρείτε το f(2).

$$\int_0^2\left[xf''(x)+3f'(x)\right]dx=4016\Rightarrow$$
$$\int_0^2\left[xf''(x)\right]dx+3\int_0^2\left[f'(x)\right]dx=4016\Rightarrow$$
$$\int_0^2\left(x\left(f'(x)\right)'\right)dx+3\left[f(x)\right]_0^2=4016\Rightarrow$$
$$\left[xf'(x)\right]_0^2-\int_0^2\left((x)'f'(x)\right)dx+3\left(f(2)-f(0)\right)=4016\Rightarrow$$
$$2f'(2)-0-\int_0^2 f'(x)dx+3\left(f(2)+2\right)=4016\Rightarrow$$
$$2\cdot 0-\left[f(x)\right]_0^2+3f(2)+6=4016\Rightarrow$$
$$-\left(f(2)-f(0)\right)+3f(2)=4010\Rightarrow$$
$$-2+2f(2)=4010\Rightarrow f(2)=2006$$

14.80

Η συνάρτηση f έχει συνεχή δεύτερη παράγωγο με f(π) = 1 και $\int_0^\pi\left(4f(x)+f''(x)\right)\eta\mu 2x dx=2$. Να αποδείξετε ότι f(0) = 2.

$$\int_0^\pi\left(4f(x)+f''(x)\right)\eta\mu 2x dx=2$$
$$\int_0^\pi\left(4f(x)\eta\mu 2x+f''(x)\eta\mu 2x\right)dx=2\Leftrightarrow$$
$$\int_0^\pi\left(4f(x)\eta\mu 2x-2f'(x)\sigma\upsilon\nu 2x+2f'(x)\sigma\upsilon\nu 2x+f''(x)\eta\mu 2x\right)dx=2\Leftrightarrow$$
$$\int_0^\pi\left(4f(x)\eta\mu 2x-2f'(x)\sigma\upsilon\nu 2x\right)dx+\int_0^\pi\left(2f'(x)\sigma\upsilon\nu 2x+f''(x)\eta\mu 2x\right)dx=2\Leftrightarrow$$
$$\int_0^\pi\left(f(x)(-2\sigma\upsilon\nu 2x)'+f'(x)(-2\sigma\upsilon\nu 2x)\right)dx+\int_0^\pi\left(f'(x)(\eta\mu 2x)'+\left(f'(x)\right)'\eta\mu 2x\right)dx=2\Leftrightarrow$$
$$\int_0^\pi\left(-2f(x)\sigma\upsilon\nu 2x\right)'dx+\int_0^\pi\left(f'(x)\eta\mu 2x\right)'dx=2\Leftrightarrow$$
$$-2\left[f(x)\sigma\upsilon\nu 2x\right]_0^\pi+\left[f'(x)\eta\mu 2x\right]_0^\pi=2\Rightarrow$$
$$-2\cdot\left(f(\pi)\sigma\upsilon\nu 2\pi-f(0)\sigma\upsilon\nu 0\right)+\left(f'(\pi)\eta\mu 2\pi-f'(0)\eta\mu 0\right)=2\Rightarrow$$
$$-2\cdot\left(1\cdot 1-f(0)\cdot 1\right)+(0-0)=2\Rightarrow$$
$$-2+2f(0)=2\Rightarrow f(0)=2$$

14.81

Αν οι συναρτήσεις f και g είναι συνεχείς στο [-α,α] , η f είναι άρτια και η g περιττή, να αποδείξετε ότι: $\boxed{\int_{-\alpha}^{\alpha} \frac{f(x)}{e^{g(x)}+1}dx = \int_0^{\alpha} f(x)\,dx}$

$$\mathrm{I} = \int_{-\alpha}^{\alpha} \frac{f(x)}{e^{g(x)}+1}dx =$$

$$\Theta\acute{\varepsilon}\tau\omega \;\; x = -u \Rightarrow dx = -du$$

$$x = \alpha \Rightarrow u_2 = -\alpha$$

$$x = -\alpha \Rightarrow u_1 = \alpha$$

$$\mathrm{I} = \int_{-\alpha}^{\alpha} \frac{f(-u)}{e^{g(-u)}+1}du \overset{\substack{f\,\acute{\alpha}\rho\tau\iota\alpha \\ g\,\pi\varepsilon\rho\iota\tau\tau\acute{\eta}}}{=} \int_{-\alpha}^{\alpha} \frac{f(u)}{e^{-g(u)}+1}du = \int_{-\alpha}^{\alpha} \frac{f(u)}{\frac{1}{e^{g(u)}}+1}du = \int_{-\alpha}^{\alpha} \frac{f(u)}{\frac{1+e^{g(u)}}{e^{g(u)}}}du = \int_{-\alpha}^{\alpha} \frac{e^{g(u)}f(u)}{1+e^{g(u)}}du$$

$\acute{E}\chi o\upsilon\mu\varepsilon$:

$$\left.\begin{aligned} \mathrm{I} &= \int_{-\alpha}^{\alpha} \frac{f(x)}{e^{g(x)}+1}dx \\ \mathrm{I} &= \int_{-\alpha}^{\alpha} \frac{e^{g(x)}\cdot f(x)}{e^{g(x)}+1}dx \end{aligned}\right| \overset{+}{\Rightarrow} 2\mathrm{I} = \int_{-\alpha}^{\alpha} \left(\frac{f(x)}{e^{g(x)}+1} + \frac{e^{g(x)}f(x)}{e^{g(x)}+1} \right) dx \Rightarrow 2\mathrm{I} = \int_{-\alpha}^{\alpha} f(x)dx \Rightarrow$$

$$\overset{\acute{\alpha}\rho\tau\iota\alpha}{\Rightarrow} 2\mathrm{I} = 2\int_0^{\alpha} f(x)dx \Rightarrow \mathrm{I} = \int_0^{\alpha} f(x)dx$$

14.82

Έστω μια συνάρτηση f: [0,1] → R η οποία είναι δύο φορές παραγωγίσιμη με συνεχή δεύτερη παράγωγο στο [0,1] και f(1) = f΄(1) = 0.

α. Να αποδείξετε ότι $\boxed{\int_0^1 f(x)\,dx = \frac{1}{2}\int_0^1 x^2 f''(x)\,dx}$

$$\int_0^1 f(x)\,dx = \frac{1}{2}\int_0^1 x^2 f''(x)\,dx \Leftrightarrow$$

$$\frac{1}{2}\int_0^1 x^2 f''(x)\,dx = \frac{1}{2}\int_0^1 x^2 \left(f'(x)\right)'\,dx = \frac{1}{2}\left(\left[x^2 f'(x)\right]_0^1 - \int_0^1 \left(x^2\right)' f'(x)\,dx \right) =$$

$$= \frac{1}{2}\left(0 - 0 - 2\int_0^1 x f'(x)\,dx\right) = -\int_0^1 x f'(x)\,dx = -\left(\left[x f(x)\right]_0^1 - \int_0^1 (x)' f(x)\,dx \right) =$$

$$= -\left(0 - 0 - \int_0^1 f(x)\,dx\right) = \int_0^1 f(x)\,dx$$

β. Αν $f''(x)=\dfrac{6}{\sqrt{x^3+1}}$, να υπολογίσετε το ολοκλήρωμα $\int_0^1 f(x)\,dx$

Λόγω του (α):

$$\int_0^1 f(x)\,dx=\frac{1}{2}\int_0^1 x^2 f''(x)\,dx=\frac{1}{2}\int_0^1\left(x^2\cdot\frac{6}{\sqrt{x^3+1}}\right)dx=\int_0^1\frac{3x^2}{\sqrt{x^3+1}}\,dx=$$

Θέτω $x^3+1=u\Rightarrow 3x^2dx=du$

$x=1\Rightarrow u_2=2$

$x=0\Rightarrow u_1=1$

$$\int_0^1 f(x)\,dx=\int_1^2\frac{1}{\sqrt{u}}\,du=2\left[\sqrt{u}\right]_1^2=2\left(\sqrt{2}-1\right)$$

14.83

Να αποδείξετε ότι:

α. $\boxed{\int_0^{\frac{\pi}{2}}\frac{\eta\mu x}{\eta\mu x+\sigma\upsilon\nu x}dx=\frac{\pi}{4}}$

$$\mathrm{A}=\int_0^{\frac{\pi}{2}}\frac{\eta\mu x}{\eta\mu x+\sigma\upsilon\nu x}dx=$$

Θέτω $x=\frac{\pi}{2}-u\Rightarrow dx=-du$

$x=\frac{\pi}{2}\Rightarrow u_2=0$

$x=0\Rightarrow u_1=\frac{\pi}{2}$

$$\mathrm{A}=\int_0^{\frac{\pi}{2}}\frac{\eta\mu\left(\frac{\pi}{2}-u\right)}{\eta\mu\left(\frac{\pi}{2}-u\right)+\sigma\upsilon\nu\left(\frac{\pi}{2}-u\right)}du=\int_0^{\frac{\pi}{2}}\frac{\sigma\upsilon\nu u}{\sigma\upsilon\nu u+\eta\mu u}du=\int_0^{\frac{\pi}{2}}\frac{\sigma\upsilon\nu x}{\sigma\upsilon\nu x+\eta\mu x}dx$$

Έχουμε :

$$\left|\begin{array}{l}\mathrm{A}=\int_0^{\frac{\pi}{2}}\frac{\sigma\upsilon\nu x}{\eta\mu x+\sigma\upsilon\nu x}dx\\ \mathrm{A}=\int_0^{\frac{\pi}{2}}\frac{\eta\mu^7 x}{\eta\mu x+\sigma\upsilon\nu x}dx\end{array}\right.\overset{+}{\Rightarrow}2\mathrm{A}=\int_0^{\frac{\pi}{2}}1dx\Rightarrow 2\mathrm{A}=\frac{\pi}{2}\Rightarrow\mathrm{A}=\frac{\pi}{4}$$

β. $\boxed{\int_0^{\frac{\pi}{2}} \frac{\eta\mu^2 x+\sigma\upsilon\nu^\nu x}{1+\eta\mu^\nu x+\sigma\upsilon\nu^\nu x}dx=\frac{\pi}{4}}$

$$\mathrm{B}=\int_0^{\frac{\pi}{2}} \frac{\eta\mu^2 x+\sigma\upsilon\nu^\nu x}{1+\eta\mu^\nu x+\sigma\upsilon\nu^\nu x}dx=$$

Θέτω $x=\frac{\pi}{2}-u \Rightarrow dx=-du$

$x=\frac{\pi}{2} \Rightarrow u_2=0$

$x=0 \Rightarrow u_1=\frac{\pi}{2}$

$$\mathrm{B}=\int_0^{\frac{\pi}{2}} \frac{\eta\mu^2\left(\frac{\pi}{2}-u\right)+\sigma\upsilon\nu^\nu\left(\frac{\pi}{2}-u\right)}{1+\eta\mu^\nu\left(\frac{\pi}{2}-u\right)+\sigma\upsilon\nu^\nu\left(\frac{\pi}{2}-u\right)}du=\int_0^{\frac{\pi}{2}} \frac{\sigma\upsilon\nu^2 u+\eta\mu^\nu u}{1+\sigma\upsilon\nu^\nu u+\eta\mu^\nu u}du=\int_0^{\frac{\pi}{2}} \frac{\sigma\upsilon\nu^2 x+\eta\mu^\nu x}{1+\sigma\upsilon\nu^\nu x+\eta\mu^\nu x}dx=$$

Έχουμε:

$$\left|\begin{array}{l}\mathrm{B}=\int_0^{\frac{\pi}{2}} \frac{\eta\mu^2 x+\sigma\upsilon\nu^\nu x}{1+\eta\mu^\nu x+\sigma\upsilon\nu^\nu x}dx \\ \mathrm{B}=\int_0^{\frac{\pi}{2}} \frac{\sigma\upsilon\nu^2 x+\eta\mu^\nu x}{1+\eta\mu^\nu x+\sigma\upsilon\nu^\nu x}dx\end{array}\right. \overset{+}{\Rightarrow} 2\mathrm{B}=\int_0^{\frac{\pi}{2}} \frac{\eta\mu^2 x+\sigma\upsilon\nu^2 x+\sigma\upsilon\nu^\nu x+\eta\mu^\nu x}{1+\eta\mu^\nu x+\sigma\upsilon\nu^\nu x}dx$$

$$\Rightarrow 2\mathrm{B}=\int_0^{\frac{\pi}{2}} 1dx \Rightarrow 2\mathrm{B}=\frac{\pi}{2} \Rightarrow \mathrm{B}=\frac{\pi}{4}$$

14.84

Να υπολογίσετε το ολοκλήρωμα: $\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{1+\sigma\upsilon\nu^{\nu}x}{2+\eta\mu^{\nu}x+\sigma\upsilon\nu^{\nu}x}dx \quad , \quad \nu\in\mathbb{N}^*$

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{1+\sigma\upsilon\nu^{\nu}x}{2+\eta\mu^{\nu}x+\sigma\upsilon\nu^{\nu}x}dx$$

$\Theta\acute{\varepsilon}\tau\omega\ x=\frac{\pi}{2}-u\Rightarrow dx=-du$

$x=\frac{\pi}{2}\Rightarrow u_2=0$

$x=0\Rightarrow u_1=\frac{\pi}{2}$

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{1+\sigma\upsilon\nu^{\nu}\left(\frac{\pi}{2}-u\right)}{2+\eta\mu^{\nu}\left(\frac{\pi}{2}-u\right)+\sigma\upsilon\nu^{\nu}\left(\frac{\pi}{2}-u\right)}du=\int_0^{\frac{\pi}{2}}\frac{1+\eta\mu^{\nu}u}{2+\sigma\upsilon\nu^{\nu}u+\eta\mu^{\nu}u}du=\int_0^{\frac{\pi}{2}}\frac{1+\eta\mu^{\nu}x}{2+\sigma\upsilon\nu^{\nu}x+\eta\mu^{\nu}x}dx=$$

$\acute{E}\chi o\upsilon\mu\varepsilon:$

$$\left|\begin{array}{l}\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{1+\sigma\upsilon\nu^{\nu}x}{2+\eta\mu^{\nu}x+\sigma\upsilon\nu^{\nu}x}dx\\ \mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{1+\eta\mu^{\nu}x}{2+\eta\mu^{\nu}x+\sigma\upsilon\nu^{\nu}x}dx\end{array}\right.\overset{+}{\Rightarrow}2\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{1+\sigma\upsilon\nu^{\nu}x+1+\eta\mu^{\nu}x}{2+\eta\mu^{\nu}x+\sigma\upsilon\nu^{\nu}x}dx$$

$$\Rightarrow 2\mathrm{I}=\int_0^{\frac{\pi}{2}}1dx\Rightarrow 2\mathrm{I}=\frac{\pi}{2}\Rightarrow \mathrm{I}=\frac{\pi}{4}$$

14.85

Να υπολογίσετε τα ολοκληρώματα:

α. $\mathrm{A}=\int_0^{\frac{\pi}{3}}\frac{1}{\sigma\upsilon\nu^4x}dx$

$$\mathrm{A}=\int_0^{\frac{\pi}{3}}\frac{1}{\sigma\upsilon\nu^4x}dx=\int_0^{\frac{\pi}{3}}\frac{1}{\sigma\upsilon\nu^2x}\cdot\frac{1}{\sigma\upsilon\nu^2x}dx=\int_0^{\frac{\pi}{3}}\left(1+\varepsilon\varphi^2x\right)\cdot\left(\varepsilon\varphi x\right)'dx=$$

$$=\int_0^{\frac{\pi}{3}}\left(\varepsilon\varphi x\right)'dx+\int_0^{\frac{\pi}{3}}\varepsilon\varphi^2x\cdot\left(\varepsilon\varphi x\right)'dx=$$

$$=\left[\varepsilon\varphi x\right]_0^{\frac{\pi}{3}}+\frac{1}{3}\left[\varepsilon\varphi^3x\right]_0^{\frac{\pi}{3}}=\left(\sqrt{3}-0\right)+\frac{1}{3}\left(\left(\sqrt{3}\right)^3-0\right)=$$

$$=\sqrt{3}+\frac{1}{3}\cdot3\sqrt{3}=2\sqrt{3}$$

β. $\boxed{\mathrm{B}=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{1}{\eta\mu^4 x}dx}$

$$\mathrm{B}=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{1}{\eta\mu^4 x}dx$$

$$\mathrm{B}=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{1}{\eta\mu^4 x}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{1}{\eta\mu^2 x}\cdot\frac{1}{\eta\mu^2 x}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(1+\sigma\varphi^2 x\right)\cdot\left(-\sigma\varphi x\right)' dx=$$

$$=-\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\sigma\varphi x\right)' dx-\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sigma\varphi^2 x\cdot\left(\sigma\varphi x\right)' dx=$$

$$=-\left[\sigma\varphi x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}-\frac{1}{3}\left[\sigma\varphi^3 x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}=-\left(1-\sqrt{3}\right)-\frac{1}{3}\left(1^3-\left(\sqrt{3}\right)^3\right)=$$

$$=-1+\sqrt{3}-\frac{1}{3}\left(1-3\sqrt{3}\right)=2\sqrt{3}-\frac{4}{3}$$

14.86

α. Αν η συνάρτηση f είναι συνεχής στο [0,π] , να αποδείξετε ότι:

$\boxed{\int_0^{\pi} xf\left(\eta\mu x\right)dx=\frac{\pi}{2}\int_0^{\pi} f\left(\eta\mu x\right)dx}$

$$\int_0^{\pi} xf\left(\eta\mu x\right)dx=$$

$\Theta\acute{\varepsilon}\tau\omega\ x=\pi-u\Rightarrow dx=-du$

$x=\pi\Rightarrow u_2=0$

$x=0\Rightarrow u_1=\pi$

$$\int_0^{\pi} xf\left(\eta\mu x\right)dx=\int_0^{\pi}\left(\pi-u\right)f\left(\eta\mu\left(\pi-u\right)\right)du=\int_0^{\pi}\left(\pi-u\right)f\left(\eta\mu u\right)du\Rightarrow$$

$$\int_0^{\pi} xf\left(\eta\mu x\right)dx=\pi\int_0^{\pi} f\left(\eta\mu u\right)du-\int_0^{\pi} uf\left(\eta\mu u\right)du\Rightarrow$$

$$\int_0^{\pi} xf\left(\eta\mu x\right)dx=\pi\int_0^{\pi} f\left(\eta\mu x\right)dx-\int_0^{\pi} xf\left(\eta\mu x\right)dx\Rightarrow$$

$$2\int_0^{\pi} xf\left(\eta\mu x\right)dx=\pi\int_0^{\pi} f\left(\eta\mu x\right)dx\Rightarrow$$

$$\int_0^{\pi} xf\left(\eta\mu x\right)dx=\frac{\pi}{2}\int_0^{\pi} f\left(\eta\mu x\right)dx$$

β. Να υπολογίσετε το ολοκλήρωμα $\boxed{\mathrm{I}=\int_0^{\pi}\frac{x\eta\mu x}{3+\eta\mu^2 x}dx}$

$$\mathrm{I}=\int_0^{\pi}\frac{x\eta\mu x}{3+\eta\mu^2 x}dx \overset{(\alpha)}{=} \frac{\pi}{2}\int_0^{\pi}\frac{\eta\mu x}{3+\eta\mu^2 x}dx=\frac{\pi}{2}\int_0^{\pi}\frac{\eta\mu x}{3+1-\sigma\upsilon\nu^2 x}dx=\frac{\pi}{2}\int_0^{\pi}\frac{\eta\mu x}{4-\sigma\upsilon\nu^2 x}dx=$$

$\Theta\acute{\varepsilon}\tau\omega\ \sigma\upsilon\nu x=u\Rightarrow -\eta\mu x dx=du\Rightarrow \eta\mu x dx=-du$

$x=\pi\Rightarrow u_2=-1$

$x=0\Rightarrow u_1=1$

$$\mathrm{I}=\frac{\pi}{2}\int_{-1}^{1}\frac{1}{4-u^2}du=\frac{\pi}{2}\int_{-1}^{1}\frac{1}{4-x^2}dx=\frac{\pi}{2}\int_{-1}^{1}\frac{1}{(2-x)(2+x)}dx=$$

$$\frac{1}{(2-x)(2+x)}=\frac{\mathrm{A}}{2-x}+\frac{\mathrm{B}}{x+2}\Leftrightarrow 1=\mathrm{A}(x+2)+\mathrm{B}(2-x)\Leftrightarrow 1=\mathrm{A}x+2\mathrm{A}-\mathrm{B}x+2\mathrm{B}$$

$$\Leftrightarrow(\mathrm{A}-\mathrm{B})x+(2\mathrm{A}+2\mathrm{B}-1)=0\Leftrightarrow\begin{cases}\mathrm{A}-\mathrm{B}=0\\2\mathrm{A}+2\mathrm{B}=1\end{cases}\Leftrightarrow\begin{cases}\mathrm{B}=\frac{1}{4}\\ \mathrm{A}=\frac{1}{4}\end{cases}$$

$\mathrm{E}\pi o\mu\acute{\varepsilon}\nu\omega\varsigma$: $\mathrm{I}=\frac{\pi}{2}\int_{-1}^{1}\left(\frac{\frac{1}{4}}{2-x}+\frac{\frac{1}{4}}{x+2}\right)dx=\frac{\pi}{8}\left[-\ln|2-x|\right]_{-1}^{1}+\frac{\pi}{8}\left[\ln|x+2|\right]_{-1}^{1}=$

$$=\frac{\pi}{8}(0+\ln 3)+\frac{\pi}{8}(\ln 3-0)=\frac{\pi}{4}\ln 3$$

14.87

Δίνεται το ολοκλήρωμα $I_\nu = \int_0^{\frac{\pi}{2}} x\sigma\upsilon\nu^\nu x\, dx$.

α. Να αποδείξετε ότι: $I_\nu = \frac{\nu-1}{\nu}\cdot I_{\nu-2} - \frac{1}{\nu^2} \quad , \quad \nu \in N \quad , \nu \geq 3$

$$I_\nu = \int_0^{\frac{\pi}{2}} x\sigma\upsilon\nu^\nu x dx = \int_0^{\frac{\pi}{2}} x\sigma\upsilon\nu^{\nu-1} x \cdot \sigma\upsilon\nu x dx = \int_0^{\frac{\pi}{2}} x\sigma\upsilon\nu^{\nu-1} x \cdot (\eta\mu x)' dx$$

$$I_\nu = \left[x\sigma\upsilon\nu^{\nu-1}x\cdot\eta\mu x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(x\sigma\upsilon\nu^{\nu-1}x\right)'\cdot\eta\mu x dx$$

$$I_\nu = 0 - 0 - \int_0^{\frac{\pi}{2}} \left(\sigma\upsilon\nu^{\nu-1}x + x(\nu-1)\sigma\upsilon\nu^{\nu-2}x\cdot(\sigma\upsilon\nu x)'\right)\cdot\eta\mu x dx$$

$$I_\nu = -\int_0^{\frac{\pi}{2}} \left(\sigma\upsilon\nu^{\nu-1}x - x(\nu-1)\sigma\upsilon\nu^{\nu-2}x\cdot\eta\mu x\right)\cdot\eta\mu x dx$$

$$I_\nu = -\int_0^{\frac{\pi}{2}} \left(\eta\mu x\sigma\upsilon\nu^{\nu-1}x - x(\nu-1)\sigma\upsilon\nu^{\nu-2}x\cdot\eta\mu^2 x\right)dx$$

$$I_\nu = -\int_0^{\frac{\pi}{2}} \left(\eta\mu x\sigma\upsilon\nu^{\nu-1}x\right)dx + (\nu-1)\int_0^{\frac{\pi}{2}} \left(x\sigma\upsilon\nu^{\nu-2}x\cdot\eta\mu^2 x\right)dx$$

$$I_\nu = \int_0^{\frac{\pi}{2}} \left((\sigma\upsilon\nu x)'\sigma\upsilon\nu^{\nu-1}x\right)dx + (\nu-1)\int_0^{\frac{\pi}{2}} \left(x\sigma\upsilon\nu^{\nu-2}x\cdot\left(1-\sigma\upsilon\nu^2 x\right)\right)dx$$

$$I_\nu = \frac{1}{\nu}\left[\sigma\upsilon\nu^\nu x\right]_0^{\frac{\pi}{2}} + (\nu-1)\left(\int_0^{\frac{\pi}{2}} \left(x\sigma\upsilon\nu^{\nu-2}x\right)dx - \int_0^{\frac{\pi}{2}} \left(x\sigma\upsilon\nu^\nu x\right)dx\right)$$

$$I_\nu = \frac{1}{\nu}(0-1) + (\nu-1)\left(I_{\nu-2} - I_\nu\right)$$

$$I_\nu = -\frac{1}{\nu} + (\nu-1)\cdot I_{\nu-2} - \nu\cdot I_\nu + I_\nu$$

$$I_\nu = \frac{\nu-1}{\nu}\cdot I_{\nu-2} - \frac{1}{\nu^2}$$

β. Να υπολογίσετε το I_3.

$$I_3 = \frac{3-1}{3}\cdot I_{3-2} - \frac{1}{3^2} = \frac{2}{3}\cdot I_1 - \frac{1}{9} \overset{*}{=} \frac{2}{3}\cdot\left(\frac{\pi}{2}-1\right) - \frac{1}{9} = \frac{\pi}{3} - \frac{7}{9}$$

$$^*I_1 = \int_0^{\frac{\pi}{2}} x\sigma\upsilon\nu x\, dx = \int_0^{\frac{\pi}{2}} x(\eta\mu x)'\, dx = \left[x\eta\mu x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (x)'\,\eta\mu x\, dx$$

$$= \frac{\pi}{2} - 0 - \int_0^{\frac{\pi}{2}} \eta\mu x\, dx = \frac{\pi}{2} + \left[\sigma\upsilon\nu x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} + (0-1) = \frac{\pi}{2} - 1$$

14.88

Να υπολογίσετε τα ολοκληρώματα:

α. $\boxed{\mathrm{A} = \int_0^{\frac{\pi}{6}} \frac{12}{3\sigma\upsilon\nu x + \sigma\upsilon\nu^3 x} dx \quad , x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

$$A=\int_0^{\frac{\pi}{6}}\frac{12}{3\sigma\upsilon\nu x+\sigma\upsilon\nu^3 x}dx \overset{\sigma\upsilon\nu x\neq 0}{=} \int_0^{\frac{\pi}{6}}\frac{12\sigma\upsilon\nu x}{3\sigma\upsilon\nu^2 x+\sigma\upsilon\nu^4 x}dx=\int_0^{\frac{\pi}{6}}\frac{12\sigma\upsilon\nu x}{3\left(1-\eta\mu^2 x\right)+\left(1-\eta\mu^2 x\right)^2}dx=$$

$\Theta\acute{\varepsilon}\tau\omega\ \eta\mu x=u\Rightarrow \sigma\upsilon\nu x dx=du$

$x=\frac{\pi}{6}\Rightarrow u_2=\frac{1}{2}$

$x=0\Rightarrow u_1=0$

$$A=\int_0^{\frac{1}{2}}\frac{12}{3\left(1-u^2\right)+\left(1-u^2\right)^2}du=\int_0^{\frac{1}{2}}\frac{12}{\left(1-u^2\right)\left[3+\left(1-u^2\right)\right]}du=\int_0^{\frac{1}{2}}\frac{12}{\left(1-u^2\right)\left(4-u^2\right)}du=$$

$$=\int_0^{\frac{1}{2}}\frac{12}{(1-u)(1+u)(2-u)(2+u)}du=\int_0^{\frac{1}{2}}\frac{12}{(1-x)(1+x)(2-x)(2+x)}dx$$

$$\frac{12}{(1-x)(1+x)(2-x)(2+x)}=\frac{A}{1-x}+\frac{B}{1+x}+\frac{\Gamma}{2-x}+\frac{\Delta}{2+x}\Leftrightarrow$$

$$12=A(1+x)(2-x)(2+x)+B(1-x)(2-x)(2+x)+\Gamma(1-x)(1+x)(2+x)+\Delta(1-x)(1+x)(2-x)\Leftrightarrow$$

$$12=A\left(-x^3-x^2+4x+4\right)+B\left(x^3-x^2-4x+4\right)+\Gamma\left(-x^3-2x^2+x+2\right)+\Delta\left(x^3-2x^2-x+2\right)\Leftrightarrow$$

$$12=\left(-A+B-\Gamma+\Delta\right)x^3+\left(-A-B-2\Gamma-2\Delta\right)x^2+\left(4A-4B+\Gamma-\Delta\right)x+\left(4A+4B+2\Gamma+2\Delta\right)\Leftrightarrow$$

$$\begin{cases}-A+B-\Gamma+\Delta=0 & (1)\\ -A-B-2\Gamma-2\Delta=0 & (2)\\ 4A-4B+\Gamma-\Delta=0 & (3)\\ 4A+4B+2\Gamma+2\Delta=12 & (4)\end{cases}\Leftrightarrow\begin{cases}A=B-\Gamma+\Delta\\ -2B-\Gamma-3\Delta=0\\ \Gamma=\Delta\\ 4B-\Gamma+3\Delta=6\end{cases}\Leftrightarrow\begin{cases}A=B-\Delta+\Delta\\ -2B-\Delta-3\Delta=0\\ \Gamma=\Delta\\ 4B-\Delta+3\Delta=6\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}A=B\\ -2B-4\Delta=0\\ \Gamma=\Delta\\ 2B+\Delta=3\end{cases}\Leftrightarrow\begin{cases}A=B\\ B+2\Delta=0\\ \Gamma=\Delta\\ 2B+\Delta=3\end{cases}\Leftrightarrow\begin{cases}A=B\\ B=2\\ \Gamma=\Delta\\ \Delta=-1\end{cases}\Leftrightarrow\begin{cases}A=2\\ B=2\\ \Gamma=-1\\ \Delta=-1\end{cases}$$

$$A=\int_0^{\frac{1}{2}}\left(\frac{2}{1-x}+\frac{2}{1+x}+\frac{-1}{2-x}+\frac{-1}{2+x}\right)dx$$

$$A=\int_0^{\frac{1}{2}}\left(\frac{2}{1-x}\right)dx+\int_0^{\frac{1}{2}}\left(\frac{2}{1+x}\right)dx-\int_0^{\frac{1}{2}}\left(\frac{1}{2-x}\right)dx-\int_0^{\frac{1}{2}}\left(\frac{1}{2+x}\right)dx$$

$$A=-2\left[\ln|1-x|\right]_0^{\frac{1}{2}}+2\left[\ln|1+x|\right]_0^{\frac{1}{2}}+\left[\ln|2-x|\right]_0^{\frac{1}{2}}-\left[\ln|2+x|\right]_0^{\frac{1}{2}}$$

$$A=-2\cdot\left(\ln\frac{1}{2}-0\right)+2\cdot\left(\ln\frac{3}{2}-0\right)+\left(\ln\frac{3}{2}-\ln 2\right)-\left(\ln\frac{5}{2}-\ln 2\right)$$

$$A=-2\cdot(-\ln 2)+2\cdot(\ln 3-\ln 2)+\ln 3-2\ln 2-\ln 5+2\ln 2$$

$$A=2\ln 2+2\ln 3-2\ln 2+\ln 3-\ln 5$$

$$A=3\ln 3-\ln 5=\ln\frac{27}{5}$$

β. $\boxed{I=\int_0^2 \ln\frac{e^x+e^2}{e^x+1}dx}$

$$I=\int_0^2 \ln\frac{e^x+e^2}{e^x+1}dx$$

Θέτω $2-x=u \Rightarrow -dx=du \Rightarrow dx=-du$

$x=2 \Rightarrow u_2=0$

$x=0 \Rightarrow u_1=2$

$$I=\int_0^2 \ln\frac{e^{2-u}+e^2}{e^{2-u}+1}du \Leftrightarrow I=\int_0^2 \ln\frac{\frac{e^2}{e^u}+e^2}{\frac{e^2}{e^u}+1}du \Leftrightarrow I=\int_0^2 \ln\frac{\frac{e^2+e^2\cdot e^u}{e^u}}{\frac{e^2+e^u}{e^u}}du \Leftrightarrow$$

$$I=\int_0^2 \ln\frac{e^2+e^2\cdot e^u}{e^2+e^u}du \Leftrightarrow I=\int_0^2 \ln\frac{e^2\left(1+e^u\right)}{e^2+e^u}du \Leftrightarrow I=\int_0^2 \ln\frac{e^2\left(1+e^x\right)}{e^2+e^x}dx$$

Έχουμε :

$$\left| \begin{array}{l} I=\int_0^2 \ln\frac{e^x+e^2}{e^x+1}dx \\ I=\int_0^2 \ln\frac{e^2\left(1+e^x\right)}{e^2+e^x}dx \end{array} \right. \overset{+}{\Rightarrow} 2I=\int_0^{\frac{\pi}{2}}\left(\ln\frac{e^x+e^2}{e^x+1}+\ln\frac{e^2\left(1+e^x\right)}{e^2+e^x}\right)dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\ln\left(\frac{e^x+e^2}{e^x+1}\cdot\frac{e^2\left(1+e^x\right)}{e^2+e^x}\right)dx \Rightarrow 2I=\int_0^2 \ln e^2\,dx \Rightarrow I=2$$

14.89

Να υπολογίσετε το ολοκλήρωμα :

$$\mathrm{A}=\int_3^4 \frac{x+4}{x^2-4}dx$$

$$\mathrm{A}=\int_3^4 \frac{x+4}{x^2-4}dx=\int_3^4 \frac{x+4}{(x+2)(x-2)}dx=$$

$$\frac{x+4}{(x+2)(x-2)}=\frac{\mathrm{A}}{x+2}+\frac{\mathrm{B}}{x-2}\Leftrightarrow x+4=\mathrm{A}(x-2)+\mathrm{B}(x+2)\Leftrightarrow x+4=\mathrm{A}x-2\mathrm{A}+\mathrm{B}x+2\mathrm{B}$$

$$\Leftrightarrow(\mathrm{A}+\mathrm{B}-1)x+(-2\mathrm{A}+2\mathrm{B}-4)=0\Leftrightarrow\begin{cases}\mathrm{A}+\mathrm{B}=1\\-\mathrm{A}+\mathrm{B}=2\end{cases}\Leftrightarrow\begin{cases}\mathrm{B}=\frac{3}{2}\\ \mathrm{A}=-\frac{1}{2}\end{cases}$$

$$\mathrm{A}=\int_3^4\left(\frac{-\frac{1}{2}}{x+2}+\frac{\frac{3}{2}}{x-2}\right)dx=-\frac{1}{2}\Big[\ln|x+2|\Big]_3^4+\frac{3}{2}\Big[\ln|x-2|\Big]_3^4$$

$$\mathrm{A}=-\frac{1}{2}(\ln 6-\ln 5)+\frac{3}{2}(\ln 2-0)=\frac{3}{2}\ln 2+\frac{1}{2}\ln 5-\frac{1}{2}\ln 6$$

$$\mathrm{B}=\int_5^{12}\frac{1+\sqrt{x+4}}{x}dx$$

Ίδια με την 14.37 Γ

14.90

Να υπολογίσετε τα ολοκληρώματα:

$$\mathrm{B}=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{\eta\mu^3 x}dx$$

$$\mathrm{B}=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{\eta\mu^3 x}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{\eta\mu^2 x}\cdot\frac{1}{\eta\mu x}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(-\sigma\varphi x\right)'\cdot\frac{1}{\eta\mu x}\,dx\Rightarrow$$

$$\mathrm{B}=\left[-\frac{\sigma\varphi x}{\eta\mu x}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(-\sigma\varphi x\right)\cdot\left(\frac{1}{\eta\mu x}\right)'\,dx\Rightarrow$$

$$\mathrm{B}=-\frac{0}{1}-\left(-\frac{\sqrt{3}}{\frac{1}{2}}\right)-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(-\sigma\varphi x\right)\cdot\left(\frac{1}{\eta\mu x}\right)'\,dx\Rightarrow$$

$$\mathrm{B}=2\sqrt{3}+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sigma\varphi x\cdot\frac{(1)'\cdot\eta\mu x-1\cdot\left(\eta\mu x\right)'}{\eta\mu^2 x}\,dx\Rightarrow$$

$$\mathrm{B}=2\sqrt{3}+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\sigma\upsilon\nu x}{\eta\mu x}\cdot\frac{-\sigma\upsilon\nu x}{\eta\mu^2 x}\,dx\Rightarrow$$

$$\mathrm{B}=2\sqrt{3}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\sigma\upsilon\nu^2 x}{\eta\mu^3 x}\,dx\Rightarrow$$

$$\mathrm{B}=2\sqrt{3}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1-\eta\mu^2 x}{\eta\mu^3 x}\,dx\Rightarrow$$

$$\mathrm{B}=2\sqrt{3}-\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{\eta\mu^3 x}\,dx-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\eta\mu^2 x}{\eta\mu^3 x}\,dx\right)\Rightarrow$$

$$\mathrm{B}=2\sqrt{3}-\left(\mathrm{B}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{\eta\mu x}\,dx\right)\Rightarrow$$

$$2\mathrm{B}=2\sqrt{3}+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{\eta\mu x}\,dx\Rightarrow$$

$$*\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{\eta\mu x}\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\eta\mu x}{\eta\mu^2 x}\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\eta\mu x}{1-\sigma\upsilon\nu^2 x}\,dx=\int_{0}^{\frac{\sqrt{3}}{2}}\frac{1}{1-u^2}\,du=\int_{0}^{\frac{\sqrt{3}}{2}}\frac{1}{(1-x)(1+x)}\,dx$$

$$\left(\begin{array}{l}\Theta\acute{\varepsilon}\tau\omega\ \sigma\upsilon\nu x=u\Rightarrow\eta\mu x dx=-du\\ x=\frac{\pi}{2}\Rightarrow u_2=0\\ x=\frac{\pi}{6}\Rightarrow u_1=\frac{\sqrt{3}}{2}\end{array}\right)$$

$$\left(\begin{array}{l}\dfrac{1}{(1-x)(1+x)}=\dfrac{A}{1-x}+\dfrac{B}{1+x}\Leftrightarrow 1=A(1+x)+B(1-x)\Leftrightarrow 1=A+Ax+B-Bx\\ \Leftrightarrow (A-B)x+(A+B-1)=0\Leftrightarrow \begin{cases}A-B=0\\ A+B=1\end{cases}\Leftrightarrow \begin{cases}B=\dfrac{1}{2}\\ A=\dfrac{1}{2}\end{cases}\end{array}\right)$$

Οπότε :

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{\eta\mu x}\,dx=\int_0^{\frac{\sqrt{3}}{2}}\left(\frac{\frac{1}{2}}{1-x}+\frac{\frac{1}{2}}{1+x}\right)dx=-\frac{1}{2}\Big[\ln|1-x|\Big]_0^{\frac{\sqrt{3}}{2}}+\frac{1}{2}\Big[\ln|1+x|\Big]_0^{\frac{\sqrt{3}}{2}}=$$

$$=-\frac{1}{2}\left(\ln\left(1-\frac{\sqrt{3}}{2}\right)-0\right)+\frac{1}{2}\left(\ln\left(1+\frac{\sqrt{3}}{2}\right)-0\right)=$$

$$=\frac{1}{2}\ln\left(1+\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\ln\left(1-\frac{\sqrt{3}}{2}\right)=$$

$$=\frac{1}{2}\left(\ln\left(\frac{2+\sqrt{3}}{2}\right)-\ln\left(\frac{2-\sqrt{3}}{2}\right)\right)=$$

$$=\frac{1}{2}\ln\frac{\frac{2+\sqrt{3}}{2}}{\frac{2-\sqrt{3}}{2}}=\frac{1}{2}\ln\frac{2+\sqrt{3}}{2-\sqrt{3}}=\frac{1}{2}\ln\frac{\left(2+\sqrt{3}\right)^2}{1}=\ln\left(2+\sqrt{3}\right)$$

Τελικά: $2B=2\sqrt{3}+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{\eta\mu x}\,dx\Rightarrow 2B=2\sqrt{3}+\ln\left(2+\sqrt{3}\right)\Rightarrow$

$\Rightarrow B=\sqrt{3}+\frac{1}{2}\ln\left(2+\sqrt{3}\right)$

14.91

Δίνεται η συνάρτηση $f(x)=\dfrac{e^x-1}{e^x+1}$, $x\in R$.

α. Να αποδείξετε ότι η f αντιστρέφεται και να βρείτε την αντίστροφη συνάρτηση f^{-1}.

$$f(x)=\frac{e^x-1}{e^x+1}\Rightarrow f'(x)=\frac{(e^x-1)'(e^x+1)-(e^x-1)(e^x+1)'}{(e^x+1)^2}\Rightarrow$$

$$\Rightarrow f'(x)=\frac{e^x(e^x+1)-(e^x-1)e^x}{(e^x+1)^2}$$

$$\Rightarrow f'(x)=\frac{e^x\left[(e^x+1)-(e^x-1)\right]}{(e^x+1)^2}$$

$$\Rightarrow f'(x)=\frac{2e^x}{(e^x+1)^2}>0,\ \Rightarrow f\ \text{γν. αύξουσα}\Rightarrow f\ \text{αντιστρέφεται.}$$

Το *πεδίο ορισμού της* f^{-1} *είναι το σύνολο τιμών* $f(\mathrm{A})$ *της* f:

f *γν. αύξουσα στο* $R\Rightarrow f(\mathrm{A})=\left(\lim\limits_{x\to-\infty}f(x),\lim\limits_{x\to+\infty}f(x)\right)=(-1,1)$

$$\left\langle\begin{array}{l}\lim\limits_{x\to-\infty}f(x)=\lim\limits_{x\to-\infty}\dfrac{e^x-1}{e^x+1}=\dfrac{0-1}{0+1}=-1\\ \lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to+\infty}\dfrac{e^x-1}{e^x+1}=\lim\limits_{x\to+\infty}\dfrac{(e^x-1)'}{(e^x+1)'}=1\end{array}\right\rangle$$

$$\mathrm{f}(x)=y\Leftrightarrow\frac{e^x-1}{e^x+1}=y\Leftrightarrow(e^x+1)y=e^x-1\Leftrightarrow e^xy+y=e^x-1\Leftrightarrow$$

$$\Leftrightarrow e^xy-e^x=-y-1\Leftrightarrow e^x(y-1)=-y-1\Leftrightarrow e^x=\frac{1+y}{1-y}\quad,y\in(-1,1)\Leftrightarrow$$

$$x=\ln\frac{1+y}{1-y}\ ,y\in(-1,1)\Leftrightarrow f^{-1}(y)=\ln\frac{1+y}{1-y}\ ,y\in(-1,1)\Leftrightarrow$$

$$\Leftrightarrow f^{-1}(x)=\ln\frac{1+x}{1-x},\mathrm{x}\in(-1,1)$$

β. Να αποδείξετε ότι η εξίσωση $f^{-1}(x)=0$ έχει μοναδική ρίζα το 0.

$$f(0)=\frac{e^0-1}{e^0+1}\Leftrightarrow f(0)=0\Leftrightarrow f^{-1}(0)=0$$

Ε*ίναι* $f^{-1}(x)=0\Leftrightarrow f^{-1}(x)=f^{-1}(0)\overset{f^{-1}1-1}{\Leftrightarrow}x=0$

γ. Να υπολογίσετε το ολοκλήρωμα $\boxed{I=\int_{-\frac{1}{2}}^{\frac{1}{2}} f^{-1}(x)\,dx}$

$$I=\int_{-\frac{1}{2}}^{\frac{1}{2}} f^{-1}(x)\,dx=\int_{-\frac{1}{2}}^{\frac{1}{2}}\ln\frac{1+x}{1-x}dx=\int_{-\frac{1}{2}}^{\frac{1}{2}}(x)'\cdot\ln\frac{1+x}{1-x}dx=$$

$$=\left[x\cdot\ln\frac{1+x}{1-x}\right]_{-\frac{1}{2}}^{\frac{1}{2}}-\int_{-\frac{1}{2}}^{\frac{1}{2}}x\cdot\left(\ln\frac{1+x}{1-x}\right)'dx=$$

$$=\frac{1}{2}\ln 3-\left(-\frac{1}{2}\right)\cdot(-\ln 3)-\int_{-\frac{1}{2}}^{\frac{1}{2}}x\cdot\frac{1}{\frac{1+x}{1-x}}\cdot\left(\frac{1+x}{1-x}\right)'dx=$$

$$=0-\int_{-\frac{1}{2}}^{\frac{1}{2}}x\cdot\frac{1-x}{1+x}\cdot\frac{2}{(1-x)^2}dx=$$

$$=-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{2x}{(1+x)(1-x)}dx=-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1+x-(1-x)}{(1+x)(1-x)}dx=$$

$$=-\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1+x}{(1+x)(1-x)}dx-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1-x}{(1+x)(1-x)}dx\right)=$$

$$=-\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{1-x}dx-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{1+x}dx\right)=-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{1-x}dx+\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{1+x}dx=$$

$$=\left[\ln|1-x|\right]_{-\frac{1}{2}}^{\frac{1}{2}}+\left[\ln|1+x|\right]_{-\frac{1}{2}}^{\frac{1}{2}}=$$

$$=\ln\frac{1}{2}-\ln\frac{3}{2}+\ln\frac{3}{2}-\ln\frac{1}{2}=0$$

14.92

Αν η συνάρτηση $f:[0,1]\to(0,+\infty)$ έχει συνεχή παράγωγο, $f(0)=1$ και $f(1)=2$, να υπολογίσετε το ολοκλήρωμα $I=\int_0^1\frac{f'(x)}{f^2(x)+f(x)}dx$

$$I=\int_0^1\frac{f'(x)}{f^2(x)+f(x)}dx=$$

$$\begin{pmatrix}\text{Θέτω } f(x)=u\Rightarrow f'(x)\,dx=du\\ x=1\Rightarrow u_2=2\\ x=0\Rightarrow u_1=1\end{pmatrix}$$

$$I=\int_0^1\frac{f'(x)}{f^2(x)+f(x)}dx=\int_1^2\frac{1}{u^2+u}du=\int_1^2\frac{1}{u(u+1)}du=\int_1^2\frac{u+1-u}{u(u+1)}du=$$

$$=\int_1^2\frac{u+1}{u(u+1)}du-\int_1^2\frac{u}{u(u+1)}du=\int_1^2\frac{1}{u}du-\int_1^2\frac{1}{u+1}du=$$

$$=\left[\ln|u|\right]_1^2-\left[\ln|u+1|\right]_1^2=\ln 2-0-(\ln 3-\ln 2)=2\ln 2-\ln 3=\ln\frac{4}{3}$$

14.93

Δίνεται το ολοκλήρωμα $\mathrm{I}(\alpha)=\int_0^{\alpha}\frac{4x}{(x^2+1)(x^2+2)(x^2+3)}dx$

α. Να υπολογίσετε την τιμή Ι(α).

$$\mathrm{I}(\alpha)=\int_0^{\alpha}\frac{4x}{(x^2+1)(x^2+2)(x^2+3)}dx=$$

$$\begin{pmatrix} \Theta\acute{\varepsilon}\tau\omega\ x^2=u\Rightarrow 2xdx=du \\ x=\alpha\Rightarrow u_2=\alpha^2 \\ x=0\Rightarrow u_1=0 \end{pmatrix}$$

$$\mathrm{I}(\alpha)=\int_0^{\alpha^2}\frac{2}{(u+1)(u+2)(u+3)}du=\int_0^{\alpha^2}\frac{2}{(x+1)(x+2)(x+3)}dx=$$

$$\left(\begin{array}{l} \frac{2}{(x+1)(x+2)(x+3)}=\frac{\mathrm{A}}{x+1}+\frac{\mathrm{B}}{x+2}+\frac{\Gamma}{x+3}\Leftrightarrow 2=\mathrm{A}(x+2)(x+3)+\mathrm{B}(x+1)(x+3)+\Gamma(x+1)(x+2) \\ \Leftrightarrow 2=\mathrm{A}(x^2+5x+6)+\mathrm{B}(x^2+4x+3)+\Gamma(x^2+3x+2)\Leftrightarrow \\ \Leftrightarrow(\mathrm{A}+\mathrm{B}+\Gamma)x^2+(5\mathrm{A}+4\mathrm{B}+3\Gamma)+(6\mathrm{A}+3\mathrm{B}+2\Gamma-2)=0\Leftrightarrow \\ \Leftrightarrow\begin{cases}\mathrm{A}+\mathrm{B}+\Gamma=0\\ 5\mathrm{A}+4\mathrm{B}+3\Gamma=0\\ 6\mathrm{A}+3\mathrm{B}+2\Gamma=2\end{cases}\Leftrightarrow\begin{cases}\mathrm{A}=-\mathrm{B}-\Gamma\\ \mathrm{B}+2\Gamma=0\\ -3\mathrm{B}-4\Gamma=2\end{cases}\Leftrightarrow\begin{cases}\mathrm{A}=-\mathrm{B}-\Gamma\\ \mathrm{B}=-2\Gamma\\ \Gamma=1\end{cases}\Leftrightarrow\begin{cases}\mathrm{A}=1\\ \mathrm{B}=-2\\ \Gamma=1\end{cases} \end{array}\right)$$

$$\mathrm{I}(\alpha)=\int_0^{\alpha^2}\left(\frac{1}{x+1}+\frac{-2}{x+2}+\frac{1}{x+3}\right)dx=\left[\ln|x+1|\right]_0^{\alpha^2}-2\left[\ln|x+2|\right]_0^{\alpha^2}+\left[\ln|x+3|\right]_0^{\alpha^2}=$$

$$=\ln(\alpha^2+1)-0-2\left(\ln(\alpha^2+2)-\ln 2\right)+\left(\ln(\alpha^2+3)-\ln 3\right)=$$

$$=\ln(\alpha^2+1)+\ln(\alpha^2+3)-2\ln(\alpha^2+2)+2\ln 2-\ln 3$$

$$=\ln\frac{(\alpha^2+1)(\alpha^2+3)}{(\alpha^2+2)^2}+\ln\frac{4}{3}=\ln\frac{4(\alpha^2+1)(\alpha^2+3)}{3(\alpha^2+2)^2}$$

β. Να υπολογίσετε την το $\lim\limits_{\alpha \to +\infty} \mathrm{I}(\alpha)$.

$$\lim_{\alpha \to +\infty} \mathrm{I}(\alpha) = \lim_{\alpha \to +\infty} \ln \frac{4(\alpha^2+1)(\alpha^2+3)}{3(\alpha^2+2)^2}$$

Θέτω $u = \dfrac{4(\alpha^2+1)(\alpha^2+3)}{3(\alpha^2+2)^2}$,

είναι $\lim\limits_{\alpha \to +\infty} \dfrac{4(\alpha^2+1)(\alpha^2+3)}{3(\alpha^2+2)^2} = \lim\limits_{\alpha \to +\infty} \dfrac{4\alpha^4}{3\alpha^4} = \dfrac{4}{3}$

Άρα $\lim\limits_{\alpha \to +\infty} \mathrm{I}(\alpha) = \lim\limits_{\alpha \to +\infty} \ln \dfrac{4(\alpha^2+1)(\alpha^2+3)}{3(\alpha^2+2)^2} = \lim\limits_{u \to \frac{4}{3}} \ln u = \ln \dfrac{4}{3}$

14.94

Αν η συνάρτηση f έχει συνεχή παράγωγο στο [α,β] και για κάθε x∈[α,β] ισχύει $f'(x)=f'(\alpha+\beta-x)$, να αποδείξετε ότι:

α. f(α) + f(β) = c

$$f'(x)=f'(\alpha+\beta-x)\Leftrightarrow f'(x)=-f'(\alpha+\beta-x)\cdot(\alpha+\beta-x)'\Leftrightarrow$$
$$f'(x)=\left(-f(\alpha+\beta-x)\right)'\Leftrightarrow f(x)=-f(\alpha+\beta-x)+c\Leftrightarrow$$
$$f(x)+f(\alpha+\beta-x)=c \quad (1)$$

$$(1)\overset{x=\alpha}{\Rightarrow} f(\alpha)+f(\alpha+\beta-\alpha)=c\Rightarrow f(\alpha)+f(\beta)=c$$

β. $\dfrac{1}{\beta-\alpha}\int_\alpha^\beta f(x)dx=\dfrac{f(\alpha)+f(\beta)}{2}$

$$(1):\quad f(x)+f(\alpha+\beta-x)=c\Rightarrow\int_\alpha^\beta\left(f(x)+f(\alpha+\beta-x)\right)dx=\int_\alpha^\beta c\,dx\Rightarrow$$
$$\Rightarrow\int_\alpha^\beta f(x)dx+\int_\alpha^\beta f(\alpha+\beta-x)dx^*=c\cdot(\beta-\alpha)\Rightarrow$$

$$^*\int_\alpha^\beta f(\alpha+\beta-x)dx=\int_\alpha^\beta f(u)du=\int_\alpha^\beta f(x)dx$$
$$\left(\begin{array}{l}\Theta\acute{\varepsilon}\tau\omega\ \alpha+\beta-x=u\Rightarrow dx=-du\\ x=\beta\Rightarrow u_2=\alpha\\ x=\alpha\Rightarrow u_1=\beta\end{array}\right)$$

$$O\pi\acute{o}\tau\varepsilon:\int_\alpha^\beta f(x)dx+\int_\alpha^\beta f(\alpha+\beta-x)dx=c\cdot(\beta-\alpha)\Rightarrow\int_\alpha^\beta f(x)dx+\int_\alpha^\beta f(x)dx=c\cdot(\beta-\alpha)\Rightarrow$$
$$\Rightarrow 2\int_\alpha^\beta f(x)dx=c\cdot(\beta-\alpha)\Rightarrow\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)dx=\frac{c}{2}\overset{(\alpha)}{\Rightarrow}\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)dx=\frac{f(\alpha)+f(\beta)}{2}$$

γ. $$\frac{1}{(\beta-\alpha)^3}\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)f(x)dx=\frac{f(\alpha)+f(\beta)}{12}$$

$$\mathrm{I}=\int_{\alpha}^{\beta}(x-\alpha)(\beta-x)f(x)dx=$$

$$\begin{pmatrix} \Theta\acute{\varepsilon}\tau\omega \ x=\alpha+\beta-u \Rightarrow dx=-du \\ x=\beta \Rightarrow u_2=\alpha \\ x=\alpha \Rightarrow u_1=\beta \end{pmatrix}$$

$$\mathrm{I}=\int_{\alpha}^{\beta}(x-\alpha)(\beta-x)f(x)dx=\int_{\alpha}^{\beta}(\alpha+\beta-u-\alpha)(\beta-\alpha-\beta+u)f(\alpha+\beta-u)du=$$

$$\mathrm{I}=\int_{\alpha}^{\beta}(\beta-u)(u-\alpha)f(\alpha+\beta-u)du\overset{(\alpha)}{\Rightarrow}$$

$$\mathrm{I}=\int_{\alpha}^{\beta}(\beta-x)(x-\alpha)(c-f(x))dx\Rightarrow$$

$$\mathrm{I}=c\int_{\alpha}^{\beta}(\beta-x)(x-\alpha)dx-\int_{\alpha}^{\beta}(\beta-x)(x-\alpha)f(x)dx\Rightarrow$$

$$\mathrm{I}=c\int_{\alpha}^{\beta}(\beta-x)(x-\alpha)dx-\mathrm{I}\Rightarrow$$

$$2\mathrm{I}=c\int_{\alpha}^{\beta}(\beta-x)(x-\alpha)dx\Rightarrow$$

$$2\mathrm{I}=c\int_{\alpha}^{\beta}\left(-x^2+(\alpha+\beta)x-\alpha\beta\right)dx\Rightarrow$$

$$2\mathrm{I}=c\left[-\frac{x^3}{3}+(\alpha+\beta)\frac{x^2}{2}-\alpha\beta x\right]_{\alpha}^{\beta}\Rightarrow$$

$$2\mathrm{I}=c\left[\left(-\frac{\beta^3}{3}+(\alpha+\beta)\frac{\beta^2}{2}-\alpha\beta^2\right)-\left(-\frac{\alpha^3}{3}+(\alpha+\beta)\frac{\alpha^2}{2}-\alpha^2\beta\right)\right]\Rightarrow$$

$$2\mathrm{I}=c\left(\frac{\beta^3-3\beta^2\alpha+3\beta\alpha^2-\alpha^3}{6}\right)\Rightarrow$$

$$2\mathrm{I}=c\cdot\frac{(\beta-\alpha)^3}{6}\Rightarrow\frac{1}{(\beta-\alpha)^3}\cdot\mathrm{I}=\frac{c}{12}\Rightarrow$$

$$\Rightarrow\frac{1}{(\beta-\alpha)^3}\int_{\alpha}^{\beta}(x-\alpha)(\beta-x)f(x)dx=\frac{f(\alpha)+f(\beta)}{12}$$

14.95

Να υπολογίσετε τα ολοκληρώματα:

α. $\mathrm{A}=\int_0^1 \frac{x\ln(x+1)}{(x+1)^2}dx$

$$\begin{pmatrix} \Theta\acute{\epsilon}\tau\omega\ x+1=u \Rightarrow dx=du \\ x=1 \Rightarrow u_2=2 \\ x=0 \Rightarrow u_1=1 \end{pmatrix}$$

$$\mathrm{A}=\int_0^1 \frac{x\ln(x+1)}{(x+1)^2}dx=\int_1^2 \frac{(u-1)\ln u}{u^2}du=\int_1^2 \frac{(x-1)\ln x}{x^2}dx=\int_1^2 \frac{x\ln x-\ln x}{x^2}dx \Rightarrow$$

$$\mathrm{A}=\int_1^2 \frac{\ln x}{x}dx-\int_1^2 \frac{\ln x}{x^2}dx \Rightarrow$$

$$\mathrm{A}=\int_1^2 \ln x\cdot(\ln x)'dx-\int_1^2 \frac{1}{x^2}\cdot \ln x dx \Rightarrow$$

$$\mathrm{A}=\frac{1}{2}\int_1^2 (\ln^2 x)'dx+\int_1^2 \left(\frac{1}{x}\right)'\cdot \ln x dx \Rightarrow$$

$$\mathrm{A}=\frac{1}{2}\left[\ln^2 x\right]_1^2+\left[\frac{\ln x}{x}\right]_1^2-\int_1^2 \frac{1}{x}\cdot(\ln x)'dx \Rightarrow$$

$$\mathrm{A}=\frac{1}{2}(\ln^2 2-0)+\frac{\ln 2}{2}-0-\int_1^2 \frac{1}{x^2}dx \Rightarrow$$

$$\mathrm{A}=\frac{1}{2}\ln^2 2+\frac{1}{2}\ln 2+\left[\frac{1}{x}\right]_1^2 \Rightarrow$$

$$\mathrm{A}=\frac{1}{2}\ln^2 2+\frac{1}{2}\ln 2+\frac{1}{2}-1 \Rightarrow \mathrm{A}=\frac{1}{2}\ln^2 2+\frac{1}{2}\ln 2-\frac{1}{2}=\frac{\ln^2 2+\ln 2-1}{2}$$

β. $\boxed{\mathrm{B}=\int_{-1}^{1}\frac{1}{\left(e^{x}+1\right)\left(x^{2}+1\right)}dx}$

$$\mathrm{B}=\int_{-1}^{1}\frac{1}{\left(e^{x}+1\right)\left(x^{2}+1\right)}dx=$$

$$\left(\begin{array}{l}\Theta\acute{\varepsilon}\tau\omega\ x=-1+1-u=-u\Rightarrow dx=-du\\ x=1\Rightarrow u_{2}=-1\\ x=-1\Rightarrow u_{1}=1\end{array}\right)$$

$$\mathrm{B}=\int_{-1}^{1}\frac{1}{\left(e^{x}+1\right)\left(x^{2}+1\right)}dx=\int_{-1}^{1}\frac{1}{\left(e^{-u}+1\right)\left(\left(-u\right)^{2}+1\right)}du=\int_{-1}^{1}\frac{1}{\left(\frac{1}{e^{u}}+1\right)\left(\left(-u\right)^{2}+1\right)}du\Rightarrow$$

$$\mathrm{B}=\int_{-1}^{1}\frac{1}{\left(\frac{1+e^{u}}{e^{u}}\right)\left(u^{2}+1\right)}du\Rightarrow\mathrm{B}=\int_{-1}^{1}\frac{e^{u}}{\left(1+e^{u}\right)\left(u^{2}+1\right)}du\Rightarrow\mathrm{B}=\int_{-1}^{1}\frac{e^{x}}{\left(1+e^{x}\right)\left(x^{2}+1\right)}dx$$

Έχουμε :

$$\left|\begin{array}{l}\mathrm{B}=\int_{-1}^{1}\frac{1}{\left(e^{x}+1\right)\left(x^{2}+1\right)}dx\\ \mathrm{B}=\int_{-1}^{1}\frac{e^{x}}{\left(e^{x}+1\right)\left(x^{2}+1\right)}dx\end{array}\right.\overset{+}{\Rightarrow}2\mathrm{B}=\int_{-1}^{1}\frac{1+e^{x}}{\left(e^{x}+1\right)\left(x^{2}+1\right)}dx\Rightarrow 2\mathrm{B}=\int_{-1}^{1}\frac{1}{x^{2}+1}dx$$

$$\Rightarrow\mathrm{B}=\frac{1}{2}\int_{-1}^{1}\frac{1}{x^{2}+1}dx\overset{*}{=}\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$

$$\left\{\begin{array}{l}{*}\Theta\acute{\varepsilon}\tau\omega\ x=\varepsilon\varphi u\Rightarrow dx=\frac{1}{\sigma\upsilon\nu^{2}u}du\\ x=1\Rightarrow\varepsilon\varphi u_{2}=0\Rightarrow u_{2}=\frac{\pi}{4}\\ x=-1\Rightarrow\varepsilon\varphi u_{1}=0\Rightarrow u_{1}=-\frac{\pi}{4}\\ \int_{-1}^{1}\frac{1}{x^{2}+1}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{\varepsilon\varphi^{2}u+1}\cdot\frac{1}{\sigma\upsilon\nu^{2}u}du=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{\frac{1}{\sigma\upsilon\nu^{2}u}}\cdot\frac{1}{\sigma\upsilon\nu^{2}u}du=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}1du=\frac{\pi}{2}\end{array}\right\}$$

14.96

Ένας μαθητής στην προσπάθειά του να υπολογίσει το ολοκλήρωμα $\mathrm{I}=\int_{-1}^{1}\frac{1}{x^2+1}dx$ έθεσε $x=\frac{1}{u}$ και έγραψε:

$$\mathrm{I}=\int_{-1}^{1}\frac{1}{x^2+1}dx=\int_{-1}^{1}\frac{1}{\frac{1}{u^2}+1}\left(-\frac{1}{u^2}\right)du=-\int_{-1}^{1}\frac{1}{1+u^2}du=-\mathrm{I}$$

Άρα : $\mathrm{I}=-\mathrm{I}\Leftrightarrow 2\mathrm{I}=0\Leftrightarrow \mathrm{I}=0$

Παρατήρησε όμως ότι $\frac{1}{x^2+1}>0\ \ ,x\in[-1,1]$, *οπότε* $\int_{-1}^{1}\frac{1}{x^2+1}dx>0$ και κατέληξε σε **αντίφαση**.

α. Σε ποια από τις δύο σκέψεις έχει κάνει λάθος;

Απάντηση: Στην 1^{η} σκέψη.

Δικαιολόγηση: Αν στον υπολογισμό του $\mathrm{I}=\int_{\alpha}^{\beta}f(x)dx$ θέσουμε x=g(u) , τότε πρέπει η συνάρτηση f(g(u)) να **ορίζεται και να είναι συνεχής** στο διάστημα $(u_1\ ,\ u_2)$,όπου u_1 = g(α) και u_2 = g(β).

Στο πάνω παράδειγμα ορίσαμε: $x=g(u)=\frac{1}{u}$

$u_1=g(-1)=\frac{1}{-1}=-1$

$u_2=g(1)=\frac{1}{1}=1$

Θα έπρεπε, λοιπόν , η συνάρτηση $f(g(u))=\frac{1}{\frac{1}{u^2}+1}$ να ορίζεται και να είναι συνεχής στο (-1,1) κάτι το οποίο δεν συμβαίνει , αφού δεν ορίζεται στο 0.

β. Να υπολογίσετε το ολοκλήρωμα Ι

Απάντηση: Δείτε την άσκηση **14.95β**

14.97

Να αποδείξετε ότι $\int_1^e \left(x^{\ln x}+e^{\sqrt{\ln x}}\right)dx = e^2-1$

Θα ξεκινήσω **όχι** με το ολοκλήρωμα $\int_1^e \left(x^{\ln x}+e^{\sqrt{\ln x}}\right)dx$, **αλλά** με το $\int_1^e x^{\ln x}dx$

$$\left(\begin{array}{l} \text{Θέτω } x^{\ln x}=u \Rightarrow \ln x^{\ln x}=\ln u \Rightarrow \ln^2 x=\ln u \Rightarrow \ln x=\sqrt{\ln u} \Rightarrow \\ \Rightarrow x=e^{\sqrt{\ln u}} \Rightarrow dx=\left(e^{\sqrt{\ln u}}\right)' du \\ x=e \Rightarrow u_2=e \\ x=1 \Rightarrow u_1=1 \end{array}\right)$$

$$\int_1^e x^{\ln x}dx=\int_1^e u\left(e^{\sqrt{\ln u}}\right)' du \Rightarrow$$

$$\int_1^e x^{\ln x}dx=\left[ue^{\sqrt{\ln u}}\right]_1^e-\int_1^e (u)' e^{\sqrt{\ln u}}du \Rightarrow$$

$$\int_1^e x^{\ln x}dx=e\cdot e^{\sqrt{\ln e}}-1\cdot e^{\sqrt{\ln 1}}-\int_1^e e^{\sqrt{\ln u}}du \Rightarrow$$

$$\int_1^e x^{\ln x}dx=e^2-1-\int_1^e e^{\sqrt{\ln x}}dx \Rightarrow$$

$$\int_1^e x^{\ln x}dx+\int_1^e e^{\sqrt{\ln x}}dx=e^2-1 \Rightarrow$$

$$\int_1^e \left(x^{\ln x}+e^{\sqrt{\ln x}}\right)dx=e^2-1$$

14.99

Δίνεται η συνεχής και άρτια συνάρτηση f στο διάστημα [-1,1] και το ολοκλήρωμα:

$\mathrm{I}=\int_0^{\frac{2\pi}{\nu}} xf\left[\sigma\upsilon\nu\left(\nu x\right)\right]dx$, $\nu\in\mathbb{N}^*$.Να αποδείξετε ότι:

α. $\boxed{\mathrm{I}=\frac{\pi}{\nu}\int_0^{\frac{2\pi}{\nu}} f\left[\sigma\upsilon\nu\left(\nu x\right)\right]dx}$

$$\mathrm{I}=\int_0^{\frac{2\pi}{\nu}} xf\left[\sigma\upsilon\nu\left(\nu x\right)\right]dx=$$

$$\left(\begin{array}{l}\Theta\acute{\varepsilon}\tau\omega\ x=\frac{2\pi}{\nu}-u\Rightarrow dx=-du\\ x=\frac{2\pi}{\nu}\Rightarrow u_2=0\\ x=0\Rightarrow u_1=\frac{2\pi}{\nu}\end{array}\right)$$

$$\mathrm{I}=\int_0^{\frac{2\pi}{\nu}} xf\left[\sigma\upsilon\nu\left(\nu x\right)\right]dx=\int_0^{\frac{2\pi}{\nu}}\left(\frac{2\pi}{\nu}-u\right)f\left[\sigma\upsilon\nu\left(\nu\left(\frac{2\pi}{\nu}-u\right)\right)\right]du\Rightarrow$$

$$\mathrm{I}=\int_0^{\frac{2\pi}{\nu}}\left(\frac{2\pi}{\nu}-u\right)f\left[\sigma\upsilon\nu\left(2\pi-\nu u\right)\right]du\Rightarrow$$

$$\mathrm{I}=\int_0^{\frac{2\pi}{\nu}}\left(\frac{2\pi}{\nu}-u\right)f\left[\sigma\upsilon\nu\left(\nu u\right)\right]du\Rightarrow$$

$$\mathrm{I}=\int_0^{\frac{2\pi}{\nu}}\left(\frac{2\pi}{\nu}-x\right)f\left[\sigma\upsilon\nu\left(\nu x\right)\right]dx\Rightarrow$$

$$\mathrm{I}=\int_0^{\frac{2\pi}{\nu}}\frac{2\pi}{\nu}\cdot f\left[\sigma\upsilon\nu\left(\nu x\right)\right]dx-\int_0^{\frac{2\pi}{\nu}} xf\left[\sigma\upsilon\nu\left(\nu x\right)\right]dx\Rightarrow$$

$$\mathrm{I}=\frac{2\pi}{\nu}\int_0^{\frac{2\pi}{\nu}} f\left[\sigma\upsilon\nu\left(\nu x\right)\right]dx-\mathrm{I}\Rightarrow\mathrm{I}=\frac{\pi}{\nu}\int_0^{\frac{2\pi}{\nu}} f\left[\sigma\upsilon\nu\left(\nu x\right)\right]dx$$

β. $\boxed{\int_0^{\frac{2\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx = 2\int_0^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx}$

$$\int_0^{\frac{2\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx = 2\int_0^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx$$

$$\int_0^{\frac{2\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx =$$

$$\begin{pmatrix} \Theta\acute{\varepsilon}\tau\omega\ x = \dfrac{\pi}{\nu} - u \Rightarrow dx = -du \\ x = \dfrac{2\pi}{\nu} \Rightarrow u_2 = -\dfrac{\pi}{\nu} \\ x = 0 \Rightarrow u_1 = \dfrac{\pi}{\nu} \end{pmatrix}$$

$$\int_0^{\frac{2\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx = \int_{-\frac{\pi}{\nu}}^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu\left(\nu\left(\frac{\pi}{\nu} - u\right)\right)\right]du = \int_{-\frac{\pi}{\nu}}^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu(\pi - \nu u)\right]du =$$

$$= \int_{-\frac{\pi}{\nu}}^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu u)\right]du = \int_{-\frac{\pi}{\nu}}^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx = \int_{-\frac{\pi}{\nu}}^{0} f\left[\sigma\upsilon\nu(\nu x)\right]dx + \int_0^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx =$$

$$= \int_0^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx + \int_0^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx = 2\int_0^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx$$

γ. $\boxed{\int_0^{\frac{2\pi}{\nu}} x f\left[\sigma\upsilon\nu(\nu x)\right]dx = \frac{4\pi}{\nu}\int_0^{\frac{\pi}{2\nu}} f\left[\eta\mu(\nu x)\right]dx}$

$\Pi\alpha\acute{\iota}\rho\nu\omega\ \tau o\ \int_0^{\frac{\pi}{2\nu}} f\left[\eta\mu(\nu x)\right]dx$

$$\begin{pmatrix} \Theta\acute{\varepsilon}\tau\omega\ x = \dfrac{\pi}{2\nu} - u \Rightarrow dx = -du \\ x = \dfrac{\pi}{2\nu} \Rightarrow u_2 = 0 \\ x = 0 \Rightarrow u_1 = \dfrac{\pi}{2\nu} \end{pmatrix}$$

$$\int_0^{\frac{\pi}{2\nu}} f\left[\eta\mu(\nu x)\right]dx = \int_0^{\frac{\pi}{2\nu}} f\left[\eta\mu\left(\nu\left(\frac{\pi}{2\nu} - u\right)\right)\right]du = \int_0^{\frac{\pi}{2\nu}} f\left[\eta\mu\left(\frac{\pi}{2} - \nu u\right)\right]du =$$

$$= \int_0^{\frac{\pi}{2\nu}} f\left[\sigma\upsilon\nu(\nu u)\right]du = \int_0^{\frac{\pi}{2\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx \quad (1)$$

$$\int_0^{\frac{2\pi}{\nu}} x f\left[\sigma\upsilon\nu(\nu x)\right]dx \overset{(\alpha)}{=} \frac{\pi}{\nu}\int_0^{\frac{2\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx \overset{(\beta)}{=} \frac{\pi}{\nu}\cdot 2\cdot\int_0^{\frac{\pi}{\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx =$$

$$\overset{(\beta)}{=} \frac{\pi}{\nu}\cdot 2\cdot 2\cdot\int_0^{\frac{\pi}{2\nu}} f\left[\sigma\upsilon\nu(\nu x)\right]dx \overset{(1)}{=} \frac{4\pi}{\nu}\int_0^{\frac{\pi}{2\nu}} f\left[\eta\mu(\nu x)\right]dx$$