If, $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in 3-space, then

a. $\|u \times v\|$ Is equal to the area of the parallelogram determined by $\boldsymbol{u}$ and $\boldsymbol{v}$.

b. Find the area of triangle determined by the points $A(2,2,0), B(-1,0,0), C(0,4,0)$.

Solution

Lagrange's identity, states that

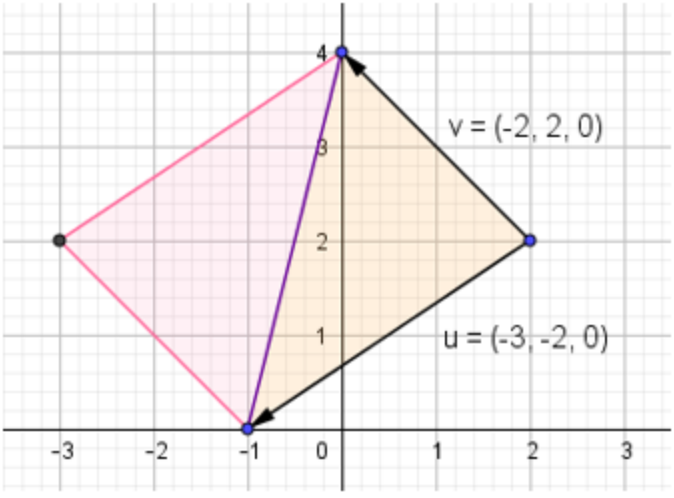
$$\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2 \quad (1)$$

If θ denotes the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, then $\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta$ (2), so (1) can be rewritten as

$$\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 \cos^2\theta \Rightarrow \ldots \|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 \sin^2\theta$$

Since $0 \le \theta \le \pi$ it follows that $\sin\theta \ge 0$, so this can be rewritten as

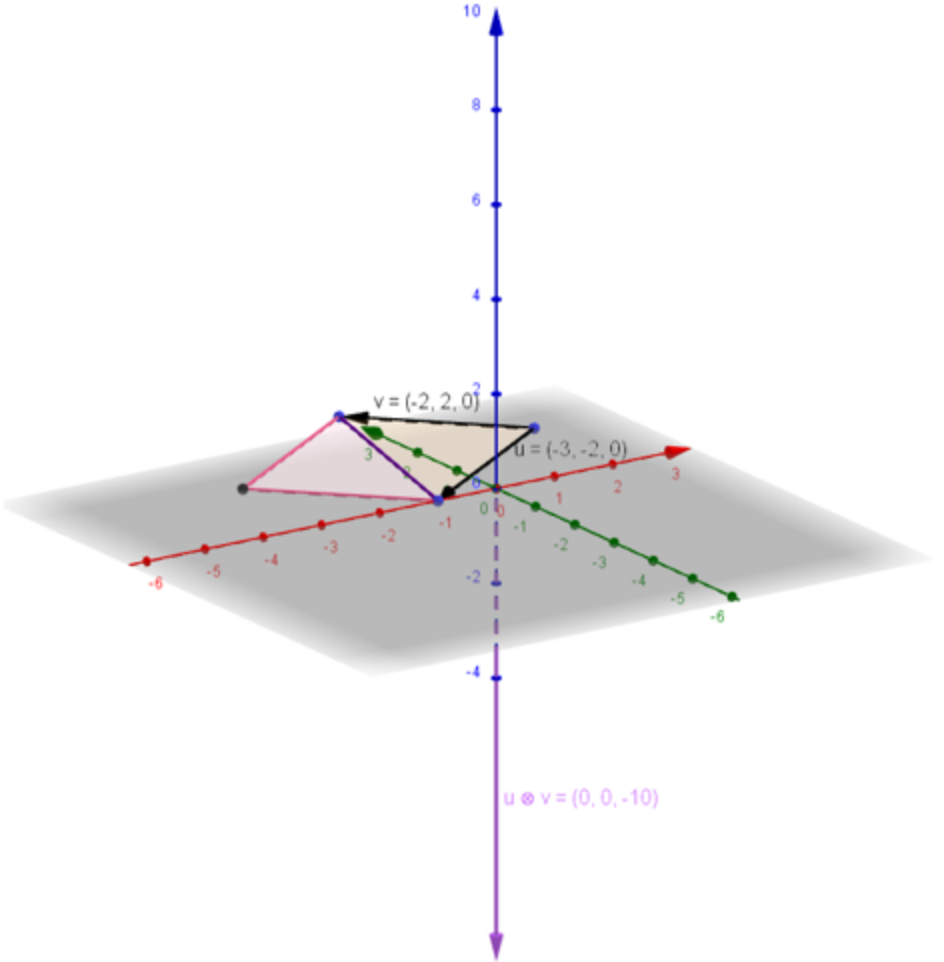
$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta$$



But $\|\boldsymbol{v}\| \sin\theta$ is the altitude of the parallelogram determined by $\boldsymbol{u}$ and $\boldsymbol{v}$. Thus the area E of this parallelogram is given by

$$E = \|u\|\|v\| \sin\theta = \|u \times v\| = \left\| \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \right\| \Rightarrow$$

$$\Rightarrow E = \|-10k\| \Rightarrow E = 10$$



b. The area A of the triangle is $\frac{1}{2}$ the area of the parallelogram and,

$$A = \frac{1}{2}E = \frac{1}{2}(10) = 5$$

page_245_cross_product.m

```
1        %  D:\...\page_245_cross_product.m
2  -     clc; clear; format compact;
3        % A = [4 -2 1];
4        % B = [1 -1 3];
5        % C = cross(A,B)
6        % dot(C,A)==0 & dot(C,B)==0
7
8  -     P1 = [2 2 0];
9  -     P2 = [-1 0 0];
10 -     P3=[0 4 0];
11 -     P1P2=[-3 -2 0];
12 -     P1P3=[-2 2 0];
13 -     c = cross(P1P2,P1P3)
14 -     n = norm(c,2)
```

Command Window

```
c =
     0     0   -10
n =
    10
```