Find a general solution of the system: $\begin{bmatrix} 3 & 4 & 1 & 2 \\ 6 & 8 & 2 & 5 \\ 9 & 12 & 3 & 10 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}$

and use that solution to find a general solution of the associated homogeneous system and a particular solution of the given system.

Solution

equ1 = 2*w + 3*x + 4*y + z == 3, equ2 = 5*w + 6*x + 8*y + 2*z == 7, equ3 = 10*w + 9*x + 12*y + 3*z == 13

The general solution of the system is:

x = 1/3 - (4*s)/3 - t/3, y=s, z=t, w=1. This solution can be expressed as the vector:

$$\begin{pmatrix} \frac{1}{3}-\frac{4}{3}s-\frac{1}{3}t \\ s \\ t \\ 1 \end{pmatrix} = s\begin{pmatrix} -\frac{4}{3} \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The corresponding homogeneous system is:

equ1 = 2*w + 3*x + 4*y + z == 0, equ2 = 5*w + 6*x + 8*y + 2*z == 0, equ3 = 10*w + 9*x + 12*y + 3*z == 0

The solution of the system:

x = 1/3 - (4*s)/3 - t/3, y=s, z=t, w=0. This solution is: $s\begin{pmatrix}-\frac{4}{3}\\ 1\\ 0\\ 0\end{pmatrix}+t\begin{pmatrix}-\frac{1}{3}\\ 0\\ 1\\ 0\end{pmatrix}$

$$\underbrace{\begin{pmatrix}\frac{1}{3}-\frac{4}{3}s-\frac{1}{3}t\\ s\\ t\\ 1\end{pmatrix}}_{\text{general solution of the system}} = \underbrace{s\begin{pmatrix}-\frac{4}{3}\\ 1\\ 0\\ 0\end{pmatrix}+t\begin{pmatrix}-\frac{1}{3}\\ 0\\ 1\\ 0\end{pmatrix}}_{\text{general solution of the homogeneous system}} + \underbrace{\begin{pmatrix}\frac{1}{3}\\ 0\\ 0\\ 1\end{pmatrix}}_{\text{a particular solution of the given system}}$$

In conclusion, this is a special case of the following general result.

The general solution of a consistent linear system $Ax = b$ can be obtained by adding any specific solution of $Ax = b$ to the general solution of $Ax = 0$.