# On a modal epistemic axiom emerging from McDermott-Doyle logics

**Costas D. Koutras**
Department of Computer Science and Technology,
University of Peloponnese,
end of Karaiskaki Street, 22100 Tripolis, Greece
ckoutras@uop.gr

**Yorgos Zikos**
Graduate Programme in Logic, Algorithms and Computation,
M.P.L.A., Dept. of Mathematics,
University of Athens
Panepistimioupolis, 157 84 Ilissia, Greece
zikos@sch.gr

June 1, 2009

**Abstract**

An important question in modal nonmonotonic logics concerns the limits of propositional definability for logics of the McDermott-Doyle family. Inspired by this technical question we define a variant of autoepistemic logic which provably corresponds to the logic of the McDermott-Doyle family that is based on the modal axiom **p5** : $\Diamond\varphi \supset (\neg\Box\varphi \supset \Box\neg\Box\varphi)$. This axiom is a natural weakening of classical negative introspection restricting its scope to *possible* facts. It closely resembles the axiom **w5** : $\varphi \supset (\neg\Box\varphi \supset \Box\neg\Box\varphi)$ which restricts the effect of negative introspection to *true* facts. We examine **p5** in the context of classical possible-worlds Kripke models, providing results for correspondence, completeness and the finite model property. We also identify the corresponding condition for **p5** in the context of neighbourhood semantics. Although rather natural epistemically, this axiom has not been investigated in classical modal epistemic reasoning, probably because its addition to **S4** gives the well-known strong modal system **S5**.

# 1 Motivation

Research in nonmonotonic reasoning has provided new paradigms for the use of modality in epistemic reasoning. Through the extensive investigation of the McDermott-Doyle family of modal nonmonotonic logics, the notion of *subnormal modal logics* (logics without axiom $\mathbf{K} : \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$, containing propositional logic and closed under rule of necessitation) has emerged; it has shown up first in [MT90] and further explored in [FMT92, ST93]. Axioms like $\mathbf{F} : \Diamond\varphi \wedge \Diamond\Box\psi \supset \Box(\Diamond\varphi \vee \psi)$ which have no obvious epistemic interpretation (although $\mathbf{F}$ has been examined from the epistemic viewpoint in the context of $\mathbf{S4} + \mathbf{F}$ [Voo93]) seem to be very important for nonmonotonic epistemic reasoning [ST94]. On the other hand, the family of autoepistemic logics invented in [Moo85] and thoroughly investigated afterwards [Kon93, MT91], introduce another paradigm of epistemic reasoning, treating modality without appealing to modal logic itself. A major argument in favour of autoepistemic logics is their clear reference to self-provability and a well motivated semantics.

The McDermott-Doyle [MD80, McD82] family of modal nonmonotonic logics is defined through the following fixpoint equation, parameterized by a monotonic modal logic $\Lambda$: assuming an initial epistemic theory $I$ of an intelligent agent, an epistemic theory $T$ is called a $\Lambda$-*expansion of* $I$ iff $T$ is consistent with $\Lambda$ and satisfies

$$T = Cn_{\Lambda}(I \cup \{\neg\Box\varphi \mid \varphi \notin T\})$$

The provability operator $Cn_{\Lambda}(S)$ of this equation is a strong one, in that it allows applications to members of the theory S, not only to $\Lambda$-theorems [MT93]. Assuming a different monotonic logic $\Lambda$, we obtain a different notion of expansion, that is, a different nonmonotonic logic. For more details (practically everything we know about this family) see [MT93].

The McDermott-Doyle extends in a certain way the modal epistemic reasoning in the nonmonotonic setting. The deficiencies initially traced in this framework, led to the discovery of autoepistemic logic [Moo85] and its many variants. This was a new idea on epistemic reasoning, one that originated directly from the world of Knowledge Representation in AI. In this framework, the reasoning is conducted in purely propositional terms: assuming a propositional modal language, keeping the proof theory of propositional logic, altering its semantics and introducing a fixpoint equation. The latter equation, reproduced below, incorporates the idea (firstly introduced in the stable sets of R. Stalnaker [Sta93]) of (i) positive introspection, in terms of an operator which adds $\Box\varphi$ to the epistemic theory $T$, whenever $\varphi \in T$ and (ii) negative introspection, adding $\neg\Box\varphi$ whenever $\varphi \notin T$. A *stable expansion* of theory $I$ is a solution of the fixpoint equation

$$T = Cn(I \cup \{\Box\varphi \mid \varphi \in T\} \cup \{\neg\Box\varphi \mid \varphi \notin T\})$$

Note that (iii) $T$ is closed under propositional provability, as $Cn$ represents the consequence operator of propositional calculus. The negative introspection part seems so perfectly natural that has been left intact in practically every variant of this logic. Nonetheless, different propositionally defined epistemic logics with a similar flavor have been defined by altering the positive introspection fragment of this equation. By assuming a positive introspection operator of the form $\{\varphi \supset \Box\varphi \mid \varphi \in T\}$ we obtain the *strict expansions* of [MT90], while by assuming $\{\varphi \equiv \Box\varphi \mid \varphi \in T\}$ instead, we obtain the *reflexive expansions* of [Sch92].

All these notions of autoepistemic reasoning are captured by the McDermott-Doyle proposal [MT93]; the first result of this stream was announced by Schwarz in [Shv90]. In a seminal result generalizing and coding all previous results in the area, Marek, Schwarz and Truszczyński proved that there exist whole intervals in the lattice of (monotonic) modal logics that generate the same nonmonotonic logic and capture the logics mentioned above [MST93]. Succintly presented: the nonmonotonic McDermott-Doyle counterparts of every logic $\Lambda$ in **5** – **KD45** captures Moore's *autoepistemic logic*, every logic $\Lambda$ in **Tw5** – **Sw5** captures *Schwarz's reflexive expansions*, every logic $\Lambda$ in **w5** – **D4w5** captures *strict expansions*, every logic $\Lambda$ in **N** – **WK** captures **N**-expansions[1].

It is a very interesting phenomenon that some logics of the McDermott-Doyle family can be written in the form of autoepistemic logic. The limits of this phenomenon is a major open issue and actually the motivating question for the results of this research note: "*The fact that these nonmonotonic logics can be defined without any reference to any modal system by a simple modification of the semantics of propositional calculus is an important advantage. It seems to be a challenging open problem, with potentially far-reaching consequences, to decide whether any other nonmonotonic logic in the McDermott-Doyle family admits similar characterization.*" [MT93, p.315]

The statement of this problem leads to a kind of *reverse logical engineering*. We define a variant of autoepistemic logic, which introduces a new modal axiom, namely one that allows us to characterize it in terms of the McDermott-Doyle definition. Our main concern is purely technical. The whole approach is entirely taken from the logician's viewpoint and we will not be concerned with the value of the logics for Knowledge Representation applications. In this investigation, we come up with a variant of axiom $\mathbf{5} : \Diamond\Box\varphi \supset \Box\varphi$ (equivalently, $\neg\Box\varphi \supset \Box\neg\Box\varphi$), the axiom of *negative introspection*. The axiom **5** states that the reasoner knows that s/he does not know a certain fact. The axiom $\mathbf{w5} : \varphi \supset (\Diamond\Box\varphi \supset \Box\varphi)$ (equivalently, $(\varphi \wedge \neg\Box\varphi) \supset \Box\neg\Box\varphi$) restricts the effect of negative introspection to true facts. The latter axiom has been examined both in classical modal logic [Seg71] (under the name **R**) and in modal nonmonotonic reasoning [Sch92, Sch95]. Here, we introduce and examine the axiom

$$\mathbf{p5} : \ \Diamond\varphi \supset (\Diamond\Box\varphi \supset \Box\varphi)$$

which restricts the effect of negative introspection to possible facts. We have chosen to name it **p5** (**p**ossibly **5**), in symmetry to the name of **w5** (**w**eak **5**). Equivalent forms of this axiom schema comprise

- $\Diamond\varphi \supset (\neg\Box\varphi \supset \Box\neg\Box\varphi)$: if something is possible and I do not know it, I know that I do not know it

[1] To keep the length of this note proportional to its contribution, we will not provide definitions for every notion mentioned. We will assume that the reader is acquainted with the notation and terminology of classical modal logic and modal nonmonotonic reasoning. Again, [BdRV01, MT93] can be consulted for details. At this point let us only mention that logic **N** is the *pure logic of necessitation*, a modal logic incorporating propositional calculus, possessing the *rule of generalization* (*rule of necessitation*) but without any axiom for modalities [FMT92]. This 'strange' logic is definitely an import of Artificial Intelligence to Modal Logic, a prime example of the so-called *subnormal modal logics*.

- $(\Diamond\varphi \wedge \Diamond\neg\varphi) \supset \Box\neg\Box\varphi$, $(\Diamond\varphi \wedge \Diamond\neg\varphi) \supset \Box\neg\Box\neg\varphi$: if everything is possible I know that I do not know anything about it.

Axiom **p5** seems to embody an interesting notion of negative introspection although, curiously enough, it has not been examined in the literature of modal epistemic reasoning [Hin62, Seg71, Len78, Len79, HC96].

This research note advocates in favour of the fact that **modal nonmonotonic logic has conceptually enriched modal logic in a variety of ways**. One of these ways, is that **axioms** or **logics** that have been **overlooked in classical epistemic reasoning come into play** through a different route. Arguably, this is also the case with axiom **p5**, which has not been examined hitherto, although it seems natural. Thus, the contributions of this paper comprise the *investigation of a natural epistemic axiom*, its *justification* in terms of modal nonmonotonic reasoning and its *evaluation* in the context of classical epistemic systems. The latter means basically the extensions of **S4**, and in section 5 a plausible partial explanation for the absence of **p5** in the literature is provided.

The structure of this paper is as follows: in Section 2 we exhibit the autoepistemic logic through which **p5** emerges. In Section 3 we examine **p5** in the realm of normal modal logics and relational semantics, providing correspondence, completeness and decidability results. In Section 4 we identify a condition corresponding to **p5** in the broader context of neighbourhood semantics. In Section 5 we discuss its value in the context of classical epistemic reasoning and finally, we discuss future research in the concluding Section 6.

A final note on notation: although the $\Diamond$ operator can be avoided, we will employ it when discussing **p5** in the context of relational models and normal modal logics. In Section 2 we avoid its usage; in general, one has to be careful when working with subnormal modal logics, as they are not closed under the equivalence substitution rule [ST93]. We believe this notational pluralism is harmless and simplifies some proofs in Section 3.2.

# 2 Narrow expansions and p5

In this section, we define a variant of autoepistemic logic and embed it in the McDermott-Doyle family. We remind the reader that we have not aimed in identifying a logic useful for KR applications; we have just focused on stretching the limits of propositional definability for McDermott-Doyle logics. To proceed with the definitions, we fix a standard monomodal language $\mathcal{L}_\Box$, assuming a countably infinite set $\Phi$ of propositional variables. The syntax is given by

$$\varphi \;::=\; p \in \Phi \mid \neg\varphi \mid \varphi_1 \supset \varphi_2 \mid \Box\varphi$$

The connectives $\wedge, \vee$ are not taken as primitive; when used, they are considered as shorthand of the equivalent expression involving $\neg, \supset$. To obtain our propositionally defined nonmonotonic logic, we modify (in a standard fashion) the classical propositional semantics.

**Definition 2.1** Let $T$ be a theory in the language $\mathcal{L}_\Box$ and $V$ a valuation of the formulae of $\mathcal{L}_\Box$ which treats $\Box\varphi$ as a propositional variable. The valuation $V$ is a *narrow autoepistemic interpretation* iff for every formula $\varphi$ in $\mathcal{L}_\Box$:

(i) $\varphi \in T$ implies that either $V(\Box\neg\varphi) = \bot$ or $V(\Box\varphi) = \top$

(ii) $\varphi \notin T$ implies that $V(\Box\varphi) = \bot$

A corresponding notion of narrow $T$-entailment emerges:

**Definition 2.2** Assume a theory $I \subseteq \mathcal{L}_\Box$; for a formula $\varphi \in \mathcal{L}_\Box$, $I \models^n_T \varphi$ iff for every narrow autoepistemic $T$-interpretation such that $V(I) = \top$ it holds that $V(\varphi) = \top$. $T$ is a *narrow expansion* of $I$ iff it satisfies the model-theoretic fixpoint equation

$$T = \{\varphi \mid I \models^n_T \varphi\}$$

**Proposition 2.3** *A consistent theory $T \subseteq \mathcal{L}_\Box$ is a narrow autoepistemic expansion of $I$ iff*

$$T = Cn(I \cup \{\neg\Box\neg\varphi \supset \Box\varphi \mid \varphi \in T\} \cup \{\neg\Box\varphi \mid \varphi \notin T\})^2 \qquad (2.3.\text{i})$$

The proof is straightforward and represents just a restatement of the semantic definition.

**Fact 2.4** *Every consistent solution of the fixpoint equation* (2.3.i) *is a stable set.*

Proof. If $T$ is a solution of (2.3.i) then obviously $T$ is closed under propositional provability and $\varphi \notin T$ implies $\neg\Box\varphi \in T$. For the positive introspection, note that $\varphi \in T$ implies $\neg\varphi \notin T$, by consistency. By the former, $\neg\Box\neg\varphi \supset \Box\varphi \in T$ and by the latter, $\neg\Box\neg\varphi \in T$. By propositional reasoning, $\Box\varphi \in T$. ∎

In order to prove that narrow expansions belong to the McDermott-Doyle family, we need the subnormal modal logic **p5**, where **p5** is the axiom $\neg\Box\neg\varphi \supset (\neg\Box\varphi \supset \Box\neg\Box\varphi)$ of restricted negative introspection.

**Theorem 2.5** *A consistent theory $T$ is a* **p5***-expansion of $I$ iff it is a narrow expansion of $I$.*

[2] The reader should easily identify the role of the positive introspection operator. In consistent expansions, it simulates the role of the rule of necessitation: note that $\varphi \in T$ implies $\neg\varphi \notin T$, and thus, $\neg\Box\neg\varphi \in T$. This route of introspective reasoning seems to be a bit 'narrow-minded' to us; hence the title for the 'narrow expansions'. Obviously the term is not well justified as we have not assessed narrow expansions from the example-driven, AI perspective. From the Modal Logic viewpoint, the positive introspection operator involved in this equation is reminiscent of the classical modal axiom of partial functionality $\mathbf{D}_c : \Diamond\varphi \supset \Box\varphi$, which has been used in dynamic logic to assert deterministic execution of programs [HKT00]. Its intuitive meaning when examined inside a state $s$, is: "*if $\varphi$ is true in an alternative state $t$, then $t$ is the unique alternative to $s$*". The validity of $\mathbf{D}_c$ corresponds to the partial functionality property: "*every world can see at most one possible world, itself or another*" [HC96, Gol92].

PROOF. ($\Rightarrow$) Assume $T$ is a **p5**-expansion of $I$. To prove that it is a narrow expansion, it suffices to show that

$$T = Cn_{\mathbf{p5}}(I \cup \{\neg\Box\varphi \mid \varphi \notin T\}) \subseteq Cn(I \cup \{\neg\Box\neg\varphi \supset \Box\varphi \mid \varphi \in T\} \cup \{\neg\Box\varphi \mid \varphi \notin T\}) \tag{2.5.i}$$

The other inclusion is straightforward since $T$ is a stable set. To prove (2.5.i), we observe that for a stable and consistent theory $T$ the set

$$S = Cn(\{\neg\Box\neg\varphi \supset \Box\varphi \mid \varphi \in T\} \cup \{\neg\Box\varphi \mid \varphi \notin T\})$$

contains every instance of 5: $\Box\varphi \vee \Box\neg\Box\varphi$ and thus, a fortiori , every instance of **p5** (by propositional reasoning).

(i) if $\varphi \in T$, by a similar argument as in Fact 2.4, $\Box\varphi \in S$.

(ii) if $\varphi \notin T$, $\neg\Box\varphi \in T$. This implies that $\neg\Box\neg\neg\Box\varphi \supset \Box\neg\Box\varphi \in S$. By the consistency assumption, $\Box\varphi \notin T$ and thus $\neg\neg\Box\varphi \notin T$. Hence, $\neg\Box\neg\neg\Box\varphi \in S$. It follows propositionally that $\Box\neg\Box\varphi \in S$.

By a similar argument as in case (i), it follows that $S$ is closed under the *rule of necessitation.*

($\Leftarrow$) Assume $T$ is a narrow expansion of $I$. It suffices to prove that

$$\{\neg\Box\varphi \mid \varphi \notin T\} \vdash_{\mathbf{p5}} \{\neg\Box\neg\varphi \supset \Box\varphi \mid \varphi \in T\} \tag{2.5.ii}$$

To this end, assume $\varphi \in T$, then $\Box\varphi \in T$ and $\neg\Box\varphi \notin T$ (by stability and consistency). Thus $\neg\Box\neg\Box\varphi \in \{\neg\Box\varphi \mid \varphi \notin T\}$. By (a propositionally equivalent variant of) **p5**: $\neg\Box\neg\Box\varphi \supset (\neg\Box\neg\varphi \supset \Box\varphi)$ and MP, (2.5.ii) follows. ∎

# 3 Relational semantics, normal modal logics and p5

In the previous section, axiom **p5** has emerged, as a basic tool for reconstructing the positive introspection part of narrow expansions, out of their negative introspection part. Yet, this axiom has a very natural epistemic interpretation, and is a very close relative of axiom **w5**. To the best of our knowledge, this is the very first time this axiom enters the modal logic literature and thus, it seems worth mentioning its basic characteristics. From the viewpoint of the technical machinery, **p5** is a simple Sahlqvist formula[3] and its analysis is fairly easy. We prove correspondence and canonicity with respect to a simple condition and then make a short Sahlqvist verification.

Although we assume that the reader is acquainted with the basics of modal logic, we wish to remind that modalities are interpreted over possible-worlds frames and models. Aiming to establish notation we reproduce the following definition and refer the interested reader to [BdRV01] for details.

[3]The celebrated *Sahlqvist theorem* identifies a large class of modal formulae which correspond to first-order conditions over modal frames; see [BdRV01, Sections 3.6 & 3.7] for a detailed presentation. We follow here the notation and analysis of this book.

**Definition 3.1** A *Kripke frame* $\mathfrak{F} = \langle W, R \rangle$ for our basic monomodal language $\mathcal{L}_\Box$ consists of a set $W$ of states (possible worlds), equipped with a binary relation $R \subseteq W \times W$.
A *Kripke model* $\mathfrak{M} = \langle W, R, V \rangle$ based on $\mathfrak{F}$, includes a valuation $V : \Phi \to \mathcal{P}(W)$ providing a truth value to every propositional variable inside each possible world. Every formula of $\mathcal{L}_\Box$ is interpreted locally, inside every state. The classical propositional connectives are interpreted in the obvious way; $\Box\varphi$ is satisfied in the state $s$ of model $\mathfrak{M}$ ($\mathfrak{M}, s \Vdash \Box\varphi$) iff

$$\forall t \in W(sRt \Rightarrow \mathfrak{M}, t \Vdash \varphi)$$

Analogously, $\Diamond\varphi$ is satisfied in the state $s$ of model $\mathfrak{M}$ ($\mathfrak{M}, s \Vdash \Diamond\varphi$) iff

$$\exists t \in W(sRt \wedge \mathfrak{M}, t \Vdash \varphi)$$

There exist *different levels of truth* in modal logic, a fact that makes the study of modal consequence relations a rich topic. We can speak about *truth in a possible world* (defined above), *validity in a model* (truth in every world of the model) and *validity in a frame* (validity in every model built on that frame).

We prove here that **p5**, semantically and proof-theoretically corresponds to the condition

$$\forall w \forall v \forall u\big((R(w,v) \wedge R(w,u) \wedge v \neq u) \supset (R(v,u) \wedge R(v,v))\big) \tag{P5}$$

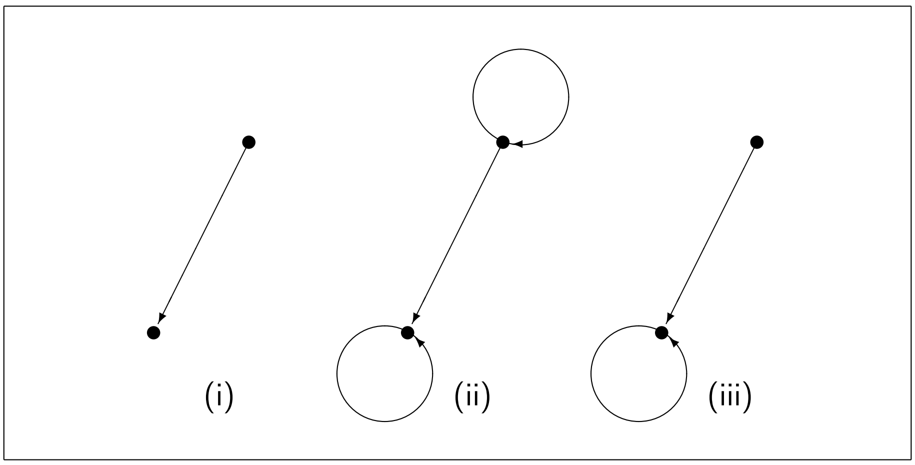
We wish to remind that axiom **5** : $\neg\Box\varphi \supset \Box\neg\Box\varphi$ corresponds to the *euclidean* property

$$\forall w \forall v \forall u\big((R(w,v) \wedge R(w,u)) \supset R(v,u)\big) \tag{Eucl}$$

while axiom **w5** : $\varphi \supset (\neg\Box\varphi \supset \Box\neg\Box\varphi)$ corresponds to the following property

$$\forall w \forall v \forall u\big((R(w,v) \wedge R(w,u) \wedge w \neq v) \supset R(u,v)\big) \tag{W5}$$

The above mentioned properties, although lead to identical situations when imposed on three different states $w \neq v \neq u$, are in general different. They are separated by the following frames:




Frame (i) is a P5 (but not a Eucl, W5) frame, frame (ii) is a W5 (but not a Eucl, P5) frame and frame (iii) satisfies all three properties.

## 3.1 Kripke Correspondence for p5

We prove here that **p5** *defines semantically* (or *corresponds to*) the frames satisfying the property (P5).

**Theorem 3.2** *Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame for $\mathcal{L}_\Box$. Then,* **p5** *is valid in $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies condition* (P5).

PROOF. ($\Rightarrow$) It suffices to show that, if (P5) does not hold in $\mathfrak{F}$, then for some $\varphi \in \mathcal{L}_\Box$, $\mathfrak{F} \nVdash \Diamond\varphi \supset (\Diamond\Box\varphi \supset \Box\varphi)$. So, since (P5) doesn't hold, there must exist $w, v, u \in W$ such that $wRv$, $wRu$ and $v \neq u$, but $\neg vRu$ or $\neg vRv$.

- Let $\neg vRu$ be the case and assume $p \in \Phi$. Concider the valuation $V$ such that $V(p) = \{v\} \cup \{z \in W \mid vRz\}$; in other words $p$ is true in $v$ and in every world $v$ sees. Now, obviously $v$ witnesses both that $\Diamond p$ and $\Diamond\Box p$ at $w$: $\langle\mathfrak{F}, V\rangle, w \Vdash \Diamond p$ and $\langle\mathfrak{F}, V\rangle, v \Vdash \Box p$ which implies $\langle\mathfrak{F}, V\rangle, w \Vdash \Diamond\Box p$. But $\Box p$ is false at $w$, since $wRu$ and $\langle\mathfrak{F}, V\rangle, u \nVdash p$ (because $u \neq v$ and $\neg vRu$). So, $\langle\mathfrak{F}, V\rangle, w \nVdash \Diamond p \supset (\Diamond\Box p \supset \Box p)$. It follows that **p5** is not valid in $\mathfrak{F}$.

- Let $\neg vRv$ be the case. In the same fashion as above, for $p \in \Phi$ we adopt the valuation $V(p) = \{u\} \cup \{z \in W \mid vRz\}$. Now, $u$ witnesses the truth of $\Diamond p$ at $w$ (since $p$ is true at $u$ and $wRu$), while $v$ witnesses the truth of $\Diamond\Box p$ at $w$. Since $v \neq u$ and $\neg vRv$, $p$ is false at $v$ and thus $\Box p$ is false at $w$. It follows again that **p5** is not valid in $\mathfrak{F}$.

($\Leftarrow$) Let $V$ be a valuation for $\mathfrak{F}$ and let $w \in W$ be such that $\langle\mathfrak{F}, V\rangle, w \Vdash \Diamond p$ i.e. there exist $v \in W$ such that

$$wRv \quad \text{and} \quad \langle\mathfrak{F}, V\rangle, v \Vdash p \tag{3.2.i}$$

Assume also, that $\langle\mathfrak{F}, V\rangle, w \Vdash \Diamond\Box p$ i.e. there exist $u \in W$ such that

$$wRu \quad \text{and} \quad \text{for all z} \in W,\ \text{if } uRz,\ \text{then } \langle\mathfrak{F}, V\rangle, z \Vdash p \tag{3.2.ii}$$

Finally, let $x \in W$ be such that $wRx$.
If $x \neq u$, then by (P5), $uRx$ and so, by (3.2.ii), $\langle\mathfrak{F}, V\rangle, x \Vdash p$.
If $x = u$, then if $u = v$, (3.2.i) leads us to $\langle\mathfrak{F}, V\rangle, x \Vdash p$. Otherwise, if $u \neq v$, then by (P5), $uRu$ and hence by (3.2.ii), $\langle\mathfrak{F}, V\rangle, u \Vdash p$ or $\langle\mathfrak{F}, V\rangle, x \Vdash p$. Consequently, $\langle\mathfrak{F}, V\rangle, w \Vdash \Box p$. ■

**Digression: the Sahlqvist identity for p5.** Following the notation and analysis of [BdRV01], it is easy for the reader to check that the universally quantified second-order transcription of **p5** (in the form $\Diamond p \supset (\Diamond\Box p \supset \Box p)$) is

$$\forall P \forall x \Big( \Big( \exists x_1 (R(x, x_1) \wedge P(x_1)) \wedge \exists x_2 \big(R(x, x_2) \wedge \forall x_4 (R(x_2, x_4) \supset P(x_4))\big)\Big) \supset \forall x_3 (R(x, x_3) \supset P(x_3)) \Big) \tag{P5s}$$

so that for every frame $\mathfrak{F}$

$$\mathfrak{F} \Vdash \mathbf{p5} \iff \mathfrak{F} \models (\mathsf{P5s})$$

Now, using the Sahlqvist algorithm, the following fact can be verified, confirming the correspondence to condition (P5). The verification is left to the reader.

**Fact 3.3** *For every Kripke frame $\mathfrak{F}$: $\mathfrak{F} \models (\mathsf{P5s}) \iff \mathfrak{F} \models (\mathsf{P5})$*

## 3.2 Completeness for P5 frames

The frame completeness theorem follows from the general pattern of *canonicity-for-a-property*: we prove that the canonical frame for any normal modal logic containing **p5** satisfies property (P5), while **p5** is valid on any class of frames satisfying (P5) (as shown in theorem 3.2). The strong frame-completeness result follows immediately.

We will not repeat the definition of the canonical frame for a normal modal logic; we follow the standard notation, as in [Gol92]. We remind that the states of the canonical frame (whose set is denoted as $W^\Lambda$) are maximal $\Lambda$-consistent sets ($\Lambda$-MCSs) and we reserve uppercase Greek letters $\Gamma, \Delta, \Xi$ for denoting $\Lambda$-MCSs. The standard definition of the accessibility relation $R^\Lambda$ between states in the canonical frame is: $\Gamma R^\Lambda \Delta$ iff for every $\varphi \in \mathcal{L}_\Box$ ($\Box\varphi \in \Gamma \Rightarrow \varphi \in \Delta$) iff for every $\varphi \in \mathcal{L}_\Box$ ($\varphi \in \Delta \Rightarrow \Diamond\varphi \in \Gamma$).

**Theorem 3.4** *Let $\Lambda$ be a normal modal logic containing* **p5**. *Then, its canonical frame satisfies* (P5), *that is*

$$\forall \Gamma, \Delta, \Xi \in W^\Lambda \big((\Gamma R^\Lambda \Delta \wedge \Gamma R^\Lambda \Xi \wedge \Delta \neq \Xi) \Rightarrow (\Delta R^\Lambda \Xi \wedge \Delta R^\Lambda \Delta)\big)$$

Proof. Let $\Gamma, \Delta, \Xi \in W^\Lambda$ be such that

$$\Gamma R^\Lambda \Delta, \quad \Gamma R^\Lambda \Xi \quad \text{and} \quad \Delta \neq \Xi \tag{3.4.i}$$

So, for all $\varphi \in \mathcal{L}_\Box$,

$$\varphi \in \Delta \Rightarrow \Diamond\varphi \in \Gamma \quad \text{and} \quad \Box\varphi \in \Gamma \Rightarrow \varphi \in \Delta \tag{3.4.ii}$$

$$\varphi \in \Xi \Rightarrow \Diamond\varphi \in \Gamma \quad \text{and} \quad \Box\varphi \in \Gamma \Rightarrow \varphi \in \Xi \tag{3.4.iii}$$

**(a)** It will be shown that $\Delta R^\Lambda \Delta$. Let $\varphi \in \Delta$. Since $\Delta$ and $\Xi$ are $\Lambda$-MCSs and different sets, they are orthogonal to each other: it cannot be the case that $\Delta \subset \Xi$ or $\Xi \subset \Delta$ [Gol92, Ch. 2]. Therefore, there exists a formula $\psi$ such that $\psi \in \Delta$ and $\psi \notin \Xi$ i.e. (since $\Xi$ is a $\Lambda$-MCS) $\psi \in \Delta$ and $\neg\psi \in \Xi$. But, for every $\Lambda$-MCS $\Delta$ it is known that $\varphi \wedge \psi \in \Delta \iff \varphi \in \Delta \wedge \psi \in \Delta$. So, we get $\varphi \wedge \psi \in \Delta$ i.e. by (3.4.ii),

$$\Diamond(\varphi \wedge \psi) \in \Gamma \tag{3.4.iv}$$

We also get $\neg\varphi \vee \neg\psi \in \Xi$. But $\neg\varphi \vee \neg\psi \supset \neg(\varphi \wedge \psi) \in \Lambda \subseteq \Xi$ (by propositional logic (PC)) and therefore (because $\Xi$ is closed under modus ponens (MP)) $\neg(\varphi \wedge \psi) \in \Xi$ i.e. by (3.4.iii),

$$\Diamond\neg(\varphi \wedge \psi) \in \Gamma \tag{3.4.v}$$

Furthermore, $\mathbf{p5} \in \Lambda$ which means (by taking the appropriate substitution instance),

$$\Diamond\neg(\varphi \wedge \psi) \supset (\Diamond(\varphi \wedge \psi) \supset \Box\Diamond(\varphi \wedge \psi)) \in \Lambda \subseteq \mathsf{\Gamma}$$

Consequently, $\Box\Diamond(\varphi \wedge \psi) \in \mathsf{\Gamma}$ (by (3.4.iv), (3.4.v) and by the fact that $\mathsf{\Gamma}$ is closed under (MP)). Hence, by (3.4.ii), $\Diamond(\varphi \wedge \psi) \in \mathsf{\Delta}$. We know that $\Diamond(\varphi \wedge \psi) \supset \Diamond\varphi \wedge \Diamond\psi$ is a theorem of every normal modal logic; it follows that $\Diamond\varphi \wedge \Diamond\psi \in \mathsf{\Delta}$ and thus $\Diamond\varphi \in \mathsf{\Delta}$. So, it has been proved that for any formula $\varphi$, $\varphi \in \mathsf{\Delta} \Rightarrow \Diamond\varphi \in \mathsf{\Delta}$, which means by definition, $\mathsf{\Delta} R^{\Lambda} \mathsf{\Delta}$.
**(b)** It remains to show that $\mathsf{\Delta} R^{\Lambda} \mathsf{\Xi}$. Let $\varphi \in \mathsf{\Xi}$; we have to prove that $\Diamond\varphi \in \mathsf{\Delta}$.
If $\varphi \in \mathsf{\Delta}$, then by a) and by definition, $\Diamond\varphi \in \mathsf{\Delta}$.
If $\varphi \notin \mathsf{\Delta}$, then $\neg\varphi \in \mathsf{\Delta}$ (because $\mathsf{\Delta}$ is a $\Lambda$-MCS). So, by (3.4.ii),

$$\Diamond\neg\varphi \in \mathsf{\Gamma} \tag{3.4.vi}$$

Since $\varphi \in \mathsf{\Xi}$, by (3.4.iii),

$$\Diamond\varphi \in \mathsf{\Gamma} \tag{3.4.vii}$$

But, $\mathbf{p5} \in \Lambda$. Therefore, $\Diamond\neg\varphi \supset (\Diamond\varphi \supset \Box\Diamond\varphi) \in \Lambda \subseteq \mathsf{\Gamma}$. Hence, by (3.4.vi), (3.4.vii) and by the fact that $\mathsf{\Gamma}$ is closed under (MP), $\Box\Diamond\varphi \in \mathsf{\Gamma}$. So, by (3.4.ii), $\Diamond\varphi \in \mathsf{\Delta}$.
So, it has been proved that for any formula $\varphi$, $\varphi \in \mathsf{\Xi} \Rightarrow \Diamond\varphi \in \mathsf{\Delta}$, which means by definition, $\mathsf{\Delta} R^{\Lambda} \mathsf{\Xi}$. ■

It follows, in a standard fashion that:

**Theorem 3.5** *The normal modal logic* **Kp5** *is sound and complete with respect to the class of all frames satisfying* (P5)*.*

## 3.3 Finite Model Property of Kp5

The *Finite Model Property* (FMP) states that any non-theorem of a logic $\Lambda$ can be falsified in a finite $\Lambda$-model [BdRV01, Sect. 3.4][Gol92, Ch. 4]. For normal modal logics it is equivalent to the *Finite Frame Property* and it guarantees decidability for a finitely axiomatized logic. We prove the FMP for **Kp5** via a variant of the *filtration* method. This method provides a flexible tool for collapsing $\Lambda$-models to construct a falsifying model for a non-theorem of a logic. We repeat some basic definitions in the interests of self-containment; more details can be found in any modal logic textbook [BdRV01, Che80, Gol92].

**Definition 3.6** Let $\Sigma$ be a set of $\mathcal{L}_\Box$ formulas and $\mathfrak{M} = \langle W, R, V\rangle$ a model. Consider a relation $\leftrightsquigarrow_\Sigma \subseteq W \times W$ s.t.:

$$w \leftrightsquigarrow_\Sigma v \iff (\forall\varphi \in \Sigma)(\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M}, v \Vdash \varphi)$$

$\leftrightsquigarrow_\Sigma$ is an equivalence relation. We denote now by $|w|_\Sigma$ (or just $|w|$) the equivalence class $[w/\leftrightsquigarrow_\Sigma]$ of $w$ (with respect to $\leftrightsquigarrow_\Sigma$), and by $W_\Sigma$ the quotient set $[\![W/\leftrightsquigarrow_\Sigma]\!]$ of $W$ (with respect to $\leftrightsquigarrow_\Sigma$). Assuming the Axiom of Choice (AC), let $\rho : W_\Sigma \to W$ be a choice

function and $V_\Sigma : \Phi \to \mathcal{P}(W_\Sigma)$ be a valuation s.t. $V_\Sigma(p) = \{\lambda \in W_\Sigma \mid \mathfrak{M}, \rho(\lambda) \Vdash p\}$ $(\forall p \in \Phi)$.
Furthermore, consider a relation $\equiv \subseteq \Sigma \times \Sigma$ defined by

$$\varphi \equiv \psi \iff \vdash_{\mathbf{K}} \varphi \leftrightarrow \psi$$

Since $\equiv$ is an equivalence relation, we denote by $[\varphi]$ the equivalence class $[\varphi/\equiv]$ of $\varphi$ (with respect to $\equiv$), and by $I_\Sigma$ the quotient set $[\![\Sigma/\equiv]\!]$ of $\Sigma$ (with respect to $\equiv$).
Finally, we define the relation $\leftrightsquigarrow_{I_\Sigma} \subseteq W \times W$ s.t.:

$$w \leftrightsquigarrow_{I_\Sigma} v \iff (\forall \psi \in \Sigma)(\forall \varphi \in [\psi])(\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M}, v \Vdash \varphi)$$

Again, $\leftrightsquigarrow_{I_\Sigma}$ is an equivalence relation and we denote by $|w|_{I_\Sigma}$ the equivalence class $[w/\leftrightsquigarrow_{I_\Sigma}]$ of $w$ (with respect to $\leftrightsquigarrow_{I_\Sigma}$), and by $W_{I_\Sigma}$ the quotient set $[\![W/\leftrightsquigarrow_{I_\Sigma}]\!]$ of $W$ (with respect to $\leftrightsquigarrow_{I_\Sigma}$).

**Definition 3.7** Given a model $\mathfrak{M} = \langle W, R, V\rangle$ and a subformula closed set of formulas $\Sigma$, a model $\langle W_\Sigma, R^f, V_\Sigma\rangle$ is called a *filtration of* $\mathfrak{M}$ *through* $\Sigma$ iff

(i) $(\forall w, v \in W)(wRv \Rightarrow |w|R^f|v|)$

(ii) $(\forall \lambda, \mu \in W_\Sigma)\big(\lambda R^f \mu \Rightarrow (\forall \Diamond\varphi \in \Sigma)(\mathfrak{M}, \rho(\mu) \Vdash \varphi \Rightarrow \mathfrak{M}, \rho(\lambda) \Vdash \Diamond\varphi)\big)$

The well-known *Filtration Theorem* states that if $\mathfrak{M}^f = \langle W_\Sigma, R^f, V_\Sigma\rangle$ is a filtration of $\mathfrak{M}$ through a subformula closed set of formulas $\Sigma$, then $(\forall \varphi \in \Sigma)(\forall w \in W)(\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M}^f, |w| \Vdash \varphi)$. The following definition actually collects the notions (and the accompanying notation) that will allow us to collapse an infinite model to a finite one. The latter, is certified by the lemma 3.8.

**Lemma 3.8** *Let $\Sigma$ be a set of $\mathcal{L}_\Box$ formulas and $\mathfrak{M} = \langle W, R, V\rangle$ a model.*

*(i)* $(\forall w, v \in W)(w \leftrightsquigarrow_\Sigma v \iff w \leftrightsquigarrow_{I_\Sigma} v)$

*(ii)* *If $I_\Sigma$ is finite, so is $W_\Sigma$.*

Proof. (i) is obvious and left to the reader. For (ii), we define function $g : W_\Sigma \to \mathcal{P}(I_\Sigma)$ s.t. for all $\lambda \in W_\Sigma$

$$g(\lambda) = \{\kappa \in I_\Sigma \mid (\exists \varphi \in \Sigma)(\kappa = [\varphi] \wedge \mathfrak{M}, \rho(\lambda) \Vdash \varphi)\}$$

Consider now, $\lambda, \mu \in W_\Sigma$ s.t. $\lambda \neq \mu$. Since, $\rho(\lambda) \in \lambda$, $|\rho(\lambda)|_\Sigma = \lambda$, and so, $|w|_\Sigma \neq |v|_\Sigma$, where $w = \rho(\lambda)$ and $v = \rho(\mu)$. Hence, $w \not\leftrightsquigarrow_\Sigma v$, and using (i), $w \not\leftrightsquigarrow_{I_\Sigma} v$. So,

$$(\exists \psi \in \Sigma)(\exists \varphi \in [\psi])(\mathfrak{M}, w \Vdash \varphi \wedge \mathfrak{M}, v \Vdash \neg\varphi)$$

therefore, $[\varphi] \in g(\lambda)$. Suppose, for the sake of contradiction, that $[\varphi] \in g(\mu)$. Then, there would be $\chi \in \Sigma$ s.t. $[\varphi] = [\chi]$ and $\mathfrak{M}, v \Vdash \chi$, hence, $\mathfrak{M}, v \Vdash \varphi$, which is a contradiction. Consequently, $g(\lambda) \neq g(\mu)$, hence, $g$ is injective, so, $W_\Sigma \leq_c \mathcal{P}(I_\Sigma)$. Since $I_\Sigma$ is finite, so is $W_\Sigma$. ∎

The following lemma allows us to construct a finite **Kp5** model for our purposes.

**Lemma 3.9** *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a model, $\Sigma$ a subformula closed set of formulas s.t. if $\Diamond\varphi \in \Sigma$, then $\Box\varphi \in \Sigma$. Consider a relation $R^p \subseteq W_\Sigma \times W_\Sigma$ s.t. $\forall\lambda, \mu \in W_\Sigma$*

$$\lambda R^p \mu \iff (\forall\Diamond\varphi \in \Sigma)(\mathfrak{M}, \rho(\mu) \Vdash \varphi \vee \Box\varphi \vee \Box\Diamond\varphi \Rightarrow \mathfrak{M}, \rho(\lambda) \Vdash \Diamond\varphi \wedge \Diamond\Box\varphi)$$

*The model $\langle W_\Sigma, R^p, V_\Sigma\rangle$ is a filtration of $\mathfrak{M}$ through $\Sigma$ and if $R$ has property* (P5)*, so does $R^p$.*

PROOF. Clearly, by the definition of filtrations, $\langle W_\Sigma, R^p, V_\Sigma\rangle$ is a filtration of $\mathfrak{M}$ through $\Sigma$. Suppose now that $R$ has property P5. Then, by theorem 3.2, $\mathfrak{M}$ is a model for **p5**. Next, consider [4] $|w|, |v|, |u| \in W_\Sigma$ s.t. $|w|R^p|v|$, $|w|R^p|u|$ and $|v| \neq |u|$.
Assume that $\Box\varphi \in \Sigma$, i.e. $\neg\Diamond\neg\varphi \in \Sigma$. But, $\Sigma$ is subformula closed. So, $\Diamond\neg\varphi \in \Sigma$, and by definition (since $|w|R^p|v|$) $\mathfrak{M}, v \Vdash \neg\varphi \vee \Box\neg\varphi \vee \Box\Diamond\neg\varphi \Rightarrow \mathfrak{M}, w \Vdash \Diamond\neg\varphi \wedge \Diamond\Box\neg\varphi$. Therefore

$$(\forall\Box\varphi \in \Sigma)(\mathfrak{M}, w \Vdash \Box\varphi \vee \Box\Diamond\varphi \Rightarrow \mathfrak{M}, v \Vdash \varphi \wedge \Diamond\varphi \wedge \Diamond\Box\varphi) \tag{3.9.i}$$

Consider now any $\Diamond\varphi \in \Sigma$ and suppose that $\mathfrak{M}, u \Vdash \varphi \vee \Box\varphi \vee \Box\Diamond\varphi$. By definition (since $|w|R^p|u|$), $\mathfrak{M}, w \Vdash \Diamond\varphi \wedge \Diamond\Box\varphi$. Hence, since **p5** is globally true in $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \Box\varphi$. So, since $\Diamond\varphi \in \Sigma$, by definition of $\Sigma$, $\Box\varphi \in \Sigma$ and using 3.9.i we get that $\mathfrak{M}, v \Vdash \Diamond\varphi \wedge \Diamond\Box\varphi$. Consequently, $|v|R^p|u|$. Similarly, it can be proved that $|v|R^p|v|$, so $R^p$ has property (P5). ∎

The following theorem concludes this section and verifies that **Kp5** has the Finite Model Property. Decidability of **Kp5** readily follows.

**Theorem 3.10** *Logic* **Kp5** *has the finite model property with respect to the class of all models satisfying* (P5)*.*

PROOF. Let $\psi$ be a satisfiable $\mathcal{L}_\Box$ formula in a (P5)-model $\langle W, R, V\rangle$. It suffices to show that $\psi$ is satisfiable in a finite (P5)-model. Consider $\Sigma$, the smallest subformula-closed set of formulas containing $\psi$, in which also the following closure condition holds: if $\Diamond\varphi \in \Sigma$, then $\Box\varphi \in \Sigma$. Using the Filtration Theorem and Lemma 3.9, we are able to prove that $\psi$ is true in the (P5)-model $\langle W_\Sigma, R^p, V_\Sigma\rangle$. It suffices to show that $I_\Sigma$ is finite; then, Lemma 3.8-(ii) immediately applies. In general, $\Sigma$ is infinite, since for every $\Diamond\varphi \in \Sigma$ it contains also $\neg\Diamond\neg\varphi$, $\Diamond\neg\varphi$, $\neg\Diamond\neg\neg\varphi$, $\Diamond\neg\neg\varphi$, $\neg\Diamond\neg\neg\neg\varphi$ and so on. But, instead of those infinitely many formulas, $I_\Sigma$ contains for every $\Diamond\varphi \in \Sigma$, only the equivalence classes $[\neg\Diamond\neg\varphi]$, $[\Diamond\neg\varphi]$, $[\neg\Diamond\varphi]$ and $[\Diamond\varphi]$. Since the set of all subformulas of $\psi$ is finite and there are only finitely many formulas of the form $\Diamond\varphi$ in $\Sigma$, $I_\Sigma$ is finite. The proof is complete. ∎

[4] The definition of $\leftrightsquigarrow_\Sigma$ guarantees that $R^p$ and $V_\Sigma$ are independent from the choice function $\rho$. So, instead of referring to an equivalence class $\lambda \in W_\Sigma$, we will simply write $|w| \in W_\Sigma$, where $w \in W$ is any representative of class $\lambda$.

## 4 Neighbourhood Semantics and p5

In this section we identify a condition for the validity of **p5** in the context of neighbourhood semantics[5]. We are not able to provide a complete exposition of this semantics in this paper; it is described as '*the most general kind of possible-worlds semantics compatible with keeping the classical truth-table semantics for the truth-functional operators*' [HC96, page 221]. The validity of axiom **K** is no more mandatory in this semantics and this has some consequences in epistemic logics.

**Definition 4.1** A *neighbourhood frame* $\mathfrak{F} = \langle W, N\rangle$ for the basic mono- modal language $\mathcal{L}_\Box$ consists of a set $W$ of states (possible worlds), equipped with a function $N : W \to \mathcal{P}(\mathcal{P}(W))$. A *neighbourhood model* $\mathfrak{M} = \langle W, N, V\rangle$ based on $\mathfrak{F}$, includes a valuation $V : \Phi \to \mathcal{P}(W)$ providing a truth value to every propositional variable inside each possible world.
As in Kripke semantics, every formula of $\mathcal{L}_\Box$ is interpreted locally, inside every state. So, an extension $\overline{V}$ of $V$ to all $\mathcal{L}_\Box$-formulae can be defined recursively, providing a truth value to every $\mathcal{L}_\Box$-formula. If $\varphi$ is true in $w \in W$, we write $w \in \overline{V}(\varphi)$ or equivalently $\mathfrak{M}, w \Vdash^n \varphi$. In the recursive definition of $\overline{V}$ the classical propositional connectives are interpreted in the obvious way. Furthermore,

$$w \in \overline{V}(\Box\varphi) \iff \overline{V}(\varphi) \in N(w)$$

Since, $\overline{V}(\Diamond\varphi) = \overline{V}(\neg\Box\neg\varphi)$, we get analogously,

$$w \in \overline{V}(\Diamond\varphi) \iff W \setminus \overline{V}(\varphi) \notin N(w)$$

Again, as in Kripke semantics, we speak about *truth in a possible world* (defined above), *validity in a model* (truth in every world of the model) and *validity in a frame* (validity in every model built on that frame).

Consider now, the following property of function $N : W \to \mathcal{P}(\mathcal{P}(W))$

$$(\forall w \in W)(\forall X \subseteq W)$$
$$\big(W \setminus X \notin N(w) \wedge \{v \in W \mid X \notin N(v)\} \notin N(w)\big) \Rightarrow X \in N(w) \qquad (\mathsf{P5n})$$

We prove below, that this condition semantically corresponds to the axiom **p5** in the context of neighbourhood demantics.

**Theorem 4.2** *Let $\mathfrak{F} = \langle W, N\rangle$ be a neighbourhood frame for $\mathcal{L}_\Box$. Then,* **p5** *is valid in $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies condition* (P5n).

PROOF. ($\Rightarrow$) We will prove the contrapositive. Assume any neighbourhood frame $\mathfrak{F} = \langle W, N\rangle$ which does not satisfy (P5n), i.e. there are $w \in W$ and $X \subseteq W$ s.t. $W \setminus X \notin$

[5] also called *minimal model semantics* in [Che80] or *Montague semantics* in other parts of the modal logic literature.

$N(w)$, $\{v \in W \mid X \notin N(v)\} \notin N(w)$ and $X \notin N(w)$. Then, define a valuation $V$ s.t. $V(p) = X$ $(p \in \Phi)$. So, $W \setminus V(p) \notin N(w)$, $W \setminus \overline{V}(\Box p) \notin N(w)$ and $V(p) \notin N(w)$. Hence, $\mathfrak{M}, w \Vdash^n \Diamond p \wedge \Diamond\Box p$, but $\mathfrak{M}, w \nVdash^n \Box p$.
($\Leftarrow$) Suppose that $\mathfrak{F}$ satisfies condition ($\mathsf{P5n}$). Let $\mathfrak{M} = \langle W, N, V\rangle$ be any neighbourhood model based on $\mathfrak{F}$, and $w$ be any state s.t. $\mathfrak{M}, w \Vdash^n \Diamond\varphi \wedge \Diamond\Box\varphi$. Then, $W \setminus \overline{V}(\varphi) \notin N(w)$ and $W \setminus \overline{V}(\Box\varphi) \notin N(w)$, i.e. $\{v \in W \mid \overline{V}(\varphi) \notin N(v)\} \notin N(w)$. Now, condition ($\mathsf{P5n}$) can be applied. Hence $\overline{V}(\varphi) \in N(w)$, i.e. $\mathfrak{M}, w \Vdash^n \Box\varphi$. ∎

We wish to remind here that this kind of results is not captured by the Sahlqvist theorem or any of its improvements and it has an independent value.

## 5 KTp5 = KT5 = S5

We come now to examine the epistemic value of **p5** in the context of classical epistemic reasoning, as initiated by Hintikka's work in the '60s [Hin62]. The interval of modal logics that has been mainly investigated in the literature for this purpose is [**S4**, **S5**] [Len79], and it is known that **KT4w5** (**Sw5**) has been proposed as a weak alternative to **S5** [Seg71]. It seems natural to consider **p5** as another alternative to negative introspection (axiom **5**); we have already noticed that we expected that it would have appeared in the literature. A possible, plausible - yet partial - explanation is provided by the fact that when **p5** is added to **S4**, **S4** + **p5** (namely, **KT4p5**) collapses to **S5**, so nothing new is gained.

**Theorem 5.1** **KT4p5** = **KTp5** = **S5**.

PROOF. Axiom **5** entails **p5** by a straightforward propositional proof, which implies that **KTp5** $\subseteq$ **KT5** = **S5**. For the other conclusion, consider the following derivation of **5** in **KTp5**:

| | | |
|---|---|---|
| 1. | $\neg p \supset \Diamond\neg p$ | axiom **T** |
| 2. | $\Box\neg p \supset \Box\Diamond\neg p$ | $1, (RM)$ |
| 3. | $\Box\neg p \wedge \Diamond\neg p \supset \Box\Diamond\neg p$ | $2, (PC) - (MP)$ |
| 4. | $\Box\neg p \supset (\Diamond\neg p \supset \Box\Diamond\neg p)$ | $3, (PC) - (MP)$ |
| 5. | $\Box\neg p \supset (\Diamond\Box p \supset \Box p)$ | $4, (PC) - (MP)$ |
| 6. | $\Diamond p \supset (\Diamond\Box p \supset \Box p)$ | axiom **p5** |
| 7. | $\Diamond p \vee \Box\neg p \supset (\Diamond\Box p \supset \Box p)$ | $5, 6, (PC) - (MP)$ |
| 8. | $\Diamond p \vee \Box\neg p$ | tautology |
| 9. | $\Diamond\Box p \supset \Box p$ | $7, 8, (MP)$ |

In the proof above, $RM$ stands for the rule $\frac{\varphi\supset\psi}{\Box\varphi\supset\Box\psi}$ [Che80, Ch. 4], $PC, MP$ stand for Propositional Calculus and Modus Ponens respectively. So, **KTp5** = **KT5** and actually **KT4p5** = **KTp5** = **S5**. ∎

It follows that **Sw5** is properly included in **KT4p5** (= **S5**). It remains to notice that **p5** could be considered as a possible alternative for negative introspection to the doxastic logic **KD45**, commonly known as the '*logic of consistent belief*'. It is not hard to identify that **KD4p5** is a genuinely new doxastic logic.

# 6 Conclusions

In this paper we have examined an axiom of negative introspection arising from a propositionally definable McDermott-Doyle logic. Starting from the fundamental question on the limits of propositional definability in this family, we have come up with axiom **p5** which is new in the modal logic literature. Apart from its potential significance for reconstructing narrow expansions, it has a natural epistemic interpretation, despite the fact that nothing genuinely is gained by replacing **5** by **p5**, in the axiomatization of **S5**. A first direction of future research would be the assessment of the logic introduced in Section 2 for Knowledge Representation; in parallel, it is worth investigating the semantics of the underlying subnormal modal logic **p5**, along the lines devised in [FMT92].

In general, we feel that it would be worth examining autoepistemic logics by trying different variants of positive or negative introspection operators. Many new interesting notions (including axioms and logics) may emerge, or gain attention in directions unforeseen hitherto. It can be hardly denied by now that, despite many initial serious objections, modal logic itself has become richer through its involvement in AI-motivated epistemic reasoning.

On the other hand, the McDermott-Doyle family has been deeply explored. It is interesting to examine variants of this fixpoint equation, and the propositional definability of the emerging logics. Up to now, the most concrete proposal has been the provability-motivated family of *boxed* and *super-boxed expansions* of [ACP96, ACGP96] which remain largely unexplored.

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