INDEFINITE INTEGRAL

SOLVED EXCERCISES

1) Find the set of all primitive functions of f(x) where:

α) $f(x)=(x-\frac{1}{x^3})^2$, $x>0$ **β)** $f(x)=2\cos(3x+1)$.

Solution:

α) $f(x)=(x-\frac{1}{x^3})^2=x^2-\frac{2}{x^2}+\frac{1}{x^6}$, thus $\int f(x)dx=\int x^2dx-2\int\frac{1}{x^2}dx+\int\frac{1}{x^6}dx=\frac{x^3}{3}+c_1-2\int x^{-2}dx+\int x^{-6}dx=\frac{x^3}{3}+c_1+2x^{-1}+c_2-\frac{x^{-5}}{5}+c_3=\frac{x^3}{3}+\frac{2}{x}+\frac{1}{5x^5}+c$.

β) $\int 2\cos(3x+1)dx=\frac{2}{3}\sin(3x+1)+c$.

Remark: Another expression of the previous exercise is: Solve the differential equation $\frac{dy}{dx}=f(x)$ where: … etc.

2) Evaluate the integrals:

α) $\int\frac{2x^2\sqrt{x}-1}{2\sqrt{x}}dx$ **β)** $\int\frac{x+1}{x+2}dx$ **γ)** $\int(x-1)^7dx$ **δ)** $\int\frac{1}{\cos^2(2x)}dx$ **ε)** $\int\frac{1}{(x-\alpha)^\nu}dx$, $\nu\in\mathbb{N}^*$.

Solution:

α) $\int\frac{2x^2\sqrt{x}-1}{2\sqrt{x}}dx=\int\left(x^2-\frac{1}{2\sqrt{x}}\right)dx=\int x^2dx-\int\frac{1}{2\sqrt{x}}dx=\frac{x^3}{3}-\sqrt{x}+c$.

β) $\int\frac{x+1}{x+2}dx=\int\frac{x+2-1}{x+2}dx=\int\left(1-\frac{1}{x+2}\right)dx=\int 1dx-\int\frac{1}{x+2}dx=x-\ln|x+2|+c$, since $(\ln|x+2|)'=\frac{1}{x+2}(x+2)'=\frac{1}{x+2}$.

γ) We have $\left[\frac{1}{8}(x-1)^8\right]'=(x-1)^7(x-1)'=(x-1)^7$, therefore $\int(x-1)^7dx=\frac{1}{8}(x-1)^8+c$.

δ) From $(\tan 2x)'=\frac{1}{\cos^2(2x)}(2x)'=\frac{2}{\cos^2(2x)}$ we have $\int\frac{1}{\cos^2(2x)}dx=\frac{1}{2}\int\frac{2}{\cos^2(2x)}dx=\frac{1}{2}\tan(2x)+c$.

ε) i) $\nu\neq1$, then $\int\frac{1}{(x-\alpha)^\nu}dx=\int(x-\alpha)^{-\nu}dx=\frac{(x-\alpha)^{-\nu+1}}{-\nu+1}+c$.

ii) $\nu=1$, then $\int\frac{1}{x-\alpha}dx=\ln|x-\alpha|+c$.

3) Show that $\int\sqrt{x^2\pm\alpha^2}dx=\frac{x}{2}\sqrt{x^2\pm\alpha^2}\pm\frac{\alpha^2}{2}\ln\left|x+\sqrt{x^2\pm\alpha^2}\right|+c$.

Solution:

We shall prove only the form with +. In a similar manner we can prove the formula with -.

It suffices to show that the derivative of the function $\frac{x}{2}\sqrt{x^2+\alpha^2}+\frac{\alpha^2}{2}\ln\left|x+\sqrt{x^2+\alpha^2}\right|+c$ equals $\sqrt{x^2+\alpha^2}$.

Indeed:

$$\left(\frac{x}{2}\sqrt{x^2+\alpha^2}+\frac{\alpha^2}{2}\ln\left|x+\sqrt{x^2+\alpha^2}\right|\right)'=\frac{1}{2}\sqrt{x^2+\alpha^2}+\frac{x}{2}\cdot\frac{2x}{2\sqrt{x^2+\alpha^2}}+\frac{\alpha^2}{2}\cdot\frac{1+\frac{2x}{2\sqrt{x^2+\alpha^2}}}{x+\sqrt{x^2+\alpha^2}}=\frac{1}{2}\sqrt{x^2+\alpha^2}+$$

$$\frac{x^2}{2\sqrt{x^2+\alpha^2}}+\frac{\alpha^2}{2}\cdot\frac{1+\frac{x}{\sqrt{x^2+\alpha^2}}}{x^2-x^2-\alpha^2}\left(x-\sqrt{x^2+\alpha^2}\right)=\frac{1}{2}\sqrt{x^2+\alpha^2}+\frac{x^2}{2\sqrt{x^2+\alpha^2}}-\frac{1+\frac{x}{\sqrt{x^2+\alpha^2}}}{2}\left(x-\sqrt{x^2+\alpha^2}\right)=$$

$$\frac{1}{2}\sqrt{x^2+\alpha^2}+\frac{x^2}{2\sqrt{x^2+\alpha^2}}-\frac{x-\sqrt{x^2+\alpha^2}+\frac{x^2}{\sqrt{x^2+\alpha^2}}-x}{2}=\frac{1}{2}\sqrt{x^2+\alpha^2}+\frac{x^2}{2\sqrt{x^2+\alpha^2}}+\frac{\alpha^2}{2\sqrt{x^2+\alpha^2}}=\sqrt{x^2+\alpha^2}.$$

4) Find the set of all primitive functions of $f(x)=3x|x|$, $x\in\mathbf{R}$.

Solution:

Let $x\in(-\infty,0]$, then $f(x)=-3x^2$, therefore $\int -3x^2dx=-x^3+c_1$.

Let $x\in(0,+\infty)$, then $f(x)=3x^2$, therefore $\int 3x^2dx=x^3+c_2$.

But a necessary condition for $F(x)=\begin{cases}-x^3+c_1 & x\le 0\\ x^3+c_2 & x>0\end{cases}$ to be a primitive function of f is to be continuous in particular to 0.

Thus it is necessary $\lim_{x\to 0^-}F(x)=\lim_{x\to 0^+}F(x)=F(0)\Leftrightarrow c_1=c_2$. Therefore $F(x)=\begin{cases}-x^3+c & x\le 0\\ x^3+c & x>0\end{cases}$.

It is easy to prove that $F'(x)=f(x)$ for every $x\in\mathbf{R}$.

5) Consider a function f with domain the set of real numbers, such that $f''(x)=12x$ and $f(0)+f'(0)+f''(0)=8$. Show that f passes through a point $(\alpha, 10\alpha)$, where $\alpha\in\mathbf{Z}$. Find this point.

Solution:

Since $\int f''(x)dx=f'(x)+c_1$ we have that $\int 12xdx=f'(x)+c_1$, that is $f'(x)=6x^2-c_1$.

But $\int f'(x)dx=f(x)+c_2$, thus $\int(6x^2-c_1)dx=f(x)+c_2$ that is $f(x)=2x^3-c_1x-c_2$.

Since $f(0)+f'(0)+f''(0)=8\Leftrightarrow -c_2-c_1=8$, if we put $c_1=c$, then $c_2=-8-c$ and thus $f(x)=2x^3-cx+8+c$.
But $f(\alpha)=10\alpha\Leftrightarrow 2\alpha^3-c\alpha+8+c=10\alpha\Leftrightarrow 2\alpha^3-c\alpha+8+c-10\alpha=0\Leftrightarrow 2\alpha^3-2\alpha-8\alpha-c\alpha+8+c=0\Leftrightarrow 2\alpha(\alpha^2-1)-8(\alpha-1)-c(\alpha-1)=0\Leftrightarrow(\alpha-1)(2\alpha^2+2\alpha-8-c)=0$. Since 1 is a root of the latter equation, the point is (1, 10).

6) Show that $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$. Evaluate the integral: $\int \tan x dx$

Solution:

α) We have $\left(\ln|f(x)|+c\right)' = \frac{1}{f(x)} \cdot f'(x)$.

β) $\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{(\cos x)'}{\cos x}dx = -\ln|\cos x| + c$.

7) If $f(x)=\frac{2-3x}{x^3}e^{-2/x}$ and $g(x)=(\kappa+\frac{\lambda-1}{x})e^{-2/x}$, find κ, λ∈R so that g is a primitive function of f.

Solution:

A necessary and sufficient condition so that g is a primitive function of g is $g'(x)=f(x)$ for every $x\in\mathbf{R}$. Thus we have $-\frac{\lambda-1}{x^2}e^{-2/x} + (\kappa+\frac{\lambda-1}{x})e^{-2/x}\cdot\frac{2}{x^2} = \frac{2-3x}{x^3}e^{-2/x} \Leftrightarrow (-\frac{\lambda-1}{x^2}+\frac{2\kappa}{x^2}+2\cdot\frac{\lambda-1}{x^3})e^{-2/x} = \frac{2-3x}{x^3}e^{-2/x} \Leftrightarrow -\frac{\lambda-1}{x^2}+\frac{2\kappa}{x^2}+2\cdot\frac{\lambda-1}{x^3}=\frac{2-3x}{x^3} \Leftrightarrow -(\lambda-1)x+2\kappa x+2\lambda-2=2-3x \Leftrightarrow [-(\lambda-1)+2\kappa]x+2\lambda-2=-3x+2$. Since the last equation is valid for every $x\in\mathbf{R}$ we have that $\{-(\lambda-1)+2\kappa=-3$ and $2\lambda-2=2\}$ $\Leftrightarrow \{\lambda=2$ and $\kappa=-1\}$.

8) Does the function $f(x)=\begin{cases} 2x+1 & x\geq 0 \\ 2x-1 & x<0 \end{cases}$ has a primitive function F in R?

Solution:

If $x>0$ then $F(x)=x^2+x+c_1$ and if $x<0$ then $F(x)=x^2-x+c_2$.

We want F to be differentiable at 0 and consequently continuous at 0. Thus we want $\lim_{x\to0^-}F(x) = \lim_{x\to0^+}F(x) = F(0) \Leftrightarrow c_1=c_2=c$.

But $\lim_{x\to0^-}\frac{F(x)-F(0)}{x-0} = \lim_{x\to0^-}\frac{x^2-x+c-c}{x} = \lim_{x\to0^-}(x-1) = -1$, $\lim_{x\to0^+}\frac{F(x)-F(0)}{x-0} = \lim_{x\to0^+}\frac{x^2+x+c-c}{x} = \lim_{x\to0^-}(x+1) = 1$, thus there is not $F'(0)$, that is f has no a primitive function in **R**.